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THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

EDITED BY

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APRIL 1918

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THE EFFECT OF SIZE OF STIMULUS AND EXPOSURE-TIME ON RETINAL THRESHOLD[1]

By PRENTICE REEVES

In determining the least amount of radiant energy which would excite the retina sufficiently to give rise to a perception of light the writer used a direct method in this laboratory. It was found that for three observers the minimum radiation visually perceptible was 19.5×10^{-10} ergs per second, and for the writer alone 17.1×10^{-10} ergs per second.[2] This value represents the mean of several observations, and seems to agree with values found by other writers.[3] The writer used a "star" 1 mm in diameter viewed at a distance of 3 m, so that the stimulus may be considered as a distant point-source of light, and the given results must be understood to apply only to the radiation from such a source. In order to get further data on the subject the writer determined the absolute threshold for stimuli ranging from a 2-millimeter square to a 12-centimeter square viewed from a distance of 35 cm. Observations were also taken in the former experiment of the

[1] *Communication No. 68* from the Research Laboratory of the Eastman Kodak Company.

[2] *Astrophysical Journal*, **46**, 167, 1917.

[3] *Ibid.*, **44**, 124, 1916; **45**, 60, 1916; **46**, 296, 1917.

1-mm star at 1.5 m and 35 cm. The writer took several observations on each stimulus over a period of about a month, so that the results are directly comparable with his previous results.

The visual sensitometer was used to control the size and intensity of the stimulus. For stimuli from 2 mm to 3 cm square the apparatus was used as described by the writer, and for the larger stimuli a flood-lamp illuminated the white matte field in which the opal-glass window is set. The size of this field could be varied and the intensity of its illumination controlled by using neutral filters, whose densities were known, with the flood-lamp. Readings were taken on the 3-centimeter stimulus in both ways and were found to agree. In the sensitometer a Nernst glower was used, and in the flood-lamp a tungsten incandescent lamp.

In the experiment the observer sat facing the apparatus at the fixed distance of 35 cm. Before making any observations he first remained in total darkness from 15 to 30 minutes to counteract any previous conditions of the retina, as otherwise the results from the observer when he had been doing dark-room work would differ from the results obtained if he had previously been working in a well-lighted room. When the eye is fully adapted to darkness it can detect the minimum amount of radiation, and its sensibility is the highest when in that state. The threshold for dark adaptation gives the absolute photo-chemical reactivity of the retina. When the eye was fully dark-adapted, the observer then determined the least intensity at which the stimulus could be seen, and this procedure was repeated for all stimuli from the 2-mm to the 12-cm squares. Each result on each day was the average of several independent observations. Several series were also run for different-sized rectangular stimuli, and the averages secured. These results are shown in Table I.

When a point-source of light is used the radiant energy is concentrated on the retina and the portion of the retina stimulated is quite small. The diameter of the writer's pupil when fully dilated was previously determined to be 8.3 mm, and with a point-source at a definite distance and known intensity it is a simple matter to compute the least perceptible radiation. But when larger light-sources are used the computation becomes more com-

plex, and at present we must be satisfied with comparative results. In the experiment with the point-source the writer used the expression

$$\text{Least perceptible radiation} = SLM\frac{r^2}{R^2} \text{ ergs per sec.},$$

where S is the area of the stimulus in square cm; L, the brightness of stimulus in lamberts; M, the mechanical equivalent of light (Ives gives this as 1.59 ergs per second per sq. cm); r, the radius

TABLE I

VARIATION OF ABSOLUTE THRESHOLD OF THE RETINA WITH SIZE AND SHAPE OF STIMULUS

Stimulus	Log. Threshold	Threshold	Energy Entering Eye
		Ml	Ergs per sec.
1-mm star at 3 m	−2.14267	0.00720	17.1×10^{-10}
" " " 1.5 m	−2.58503	.00260	24.8×10^{-10}
" " " 35 cm	−3.61979	.00024	42.1×10^{-10}
2-mm square at 35 cm	−4.54837	0.000028290	25.3×10^{-10}
5- " " " "	−5.17881	6625	37.
1-cm " " "	−5.61744	2413	54.
2- " " " "	−5.99012	1023	91.
3- " " " "	−6.34659	450	90.7
6- " " " "	−6.58872	258	208.
12- " " " "	−6.75647	175	564.
Horizontal Divisions:			
$3\times\frac{1}{2}$-cm rectangle at 35 cm	−5.65694	0.00000220	73.9×10^{-10}
3×1- " " " "	−5.82652	149	100.
$3\times1\frac{1}{2}$- " " " "	−5.91937	120	121.
3×2- " " " "	−6.10210	79	106.
$3\times2\frac{1}{2}$- " " " "	−6.21667	61	102.
3×3- " " " "	−6.33283	46	92.7
Vertical Divisions:			
$\frac{1}{2}\times3$	−5.57873	0.00000264	88.7×10^{-10}
1×3	−5.83239	147	98.6
$1\frac{1}{2}\times3$	−5.97840	105	106.
2×3	−5.26384	54	72.5
$2\frac{1}{2}\times3$	−5.34921	45	75.5
3×3	−5.39924	40	80.6

of the pupil, and R, the distance of the eye from the stimulus. The stimuli used with the visual sensitometer are uniformly diffusing or reflecting and may be considered as made up of numerous point-sources. In using the foregoing expression for the larger

stimuli we are able to get relative results, and while not theoretically accurate they will serve the purpose of this experiment. From the results it can be seen that as the visual angle increases the threshold decreases, but that the amount of radiant energy increases. The values of energy for the largest stimuli are probably too large, owing to unavoidable difficulties of manipulation, but at least they show the tendencies of the larger visual angles.

In the second part of this investigation the writer attempted to determine how long a time was required for the point-source of light to produce a perceptible sensation. The observer sat at a distance of 3 m with his head in a headrest and his eye fixated on the stimulus. The intensity was regulated until the observer could just see the star, and an assistant then covered the star. The observer kept his fixation; the assistant uncovered the star and recorded with a stop watch the time elapsing before the observer saw the star. At first the assistant did not notify the observer when the star was uncovered, and the watch was operated silently. Next the observer was notified, and it was found that unless the interval before silent exposure was quite long, telling the observer did not affect the results, as the observer's attention was constantly fixed on the stimulus. For the just perceptible intensity of the stimulus the perception-time recorded ranged from 1 to as much as 6 seconds, although the result of several observations gave an average of 2.2 seconds for the writer. Two other observers averaged 2.1 seconds each, although the intensity of the star in each threshold was higher than for the writer. When the intensity of the star was made just perceptibly brighter, the time for detection was lowered to 1.4 seconds for the writer and 1.2 and 0.85 seconds for the other observers. In several hundred observations on the visual sensitometer in various experiments the writer has noticed that practically all stimuli that can be perceived at all will be detected within two or three seconds. It has been found that if a stimulus could not be detected after 4 seconds' exposure it was not worth while to use longer periods of time, since the numerous factors that enter, such as attention, fatigue, eye movements, etc., make results unreliable.

As the results of this determination of perception-time were rather unsatisfactory, the writer tried another method. Carefully calibrated photographic shutters were placed before the test-spot of the visual sensitometer, and the writer determined the brightness of the test-spot necessary for it to be perceived when exposed for a given time. The test-spot was taken as a 3-centimeter square, and the observations were taken at a distance of 35 cm for a range of exposures from 0.002 seconds to 4 seconds. These results are shown in Table II. Observations made for exposures of 8, 12, and

TABLE II

VARIATION OF ABSOLUTE THRESHOLD OF THE RETINA WITH TIME OF EXPOSURE OF STIMULUS

Time	Log. Threshold	Threshold	1/T
Sec.		Ml	
0.002.....	−3.44092	0.00036230	2.76
.006.....	−4.01092	9752	10.3
.011.....	−4.35	4458	22.4
.020.....	−4.62	2394	41.7
.034.....	−4.91	1228	81.5
.160.....	−5.15	706	142.
.250.....	−5.29	512	195.
.500.....	−5.45	354	282.
1.000.....	−5.58	262	382.
2.000.....	−6.11	77	1300.
4.000.....	−6.20	63	1588.

20 seconds almost invariably show a decrease of retinal sensibility and were not included in this paper. The effect of exposure on sensibility is shown in the column headed 1/T, as the reciprocal of the threshold is proportional to sensibility. It is seen that the sensibility increases quite rapidly up to 2 seconds.

The writer would like to see other experiments on this subject and further work on the effect of exposure-time on smaller stimuli. He wishes to acknowledge the assistance of Julian Blanchard and Milton Fillius in this experiment.

RESEARCH LABORATORY, EASTMAN KODAK COMPANY
January 30, 1918

ON PARALLAXES AND MOTION OF THE BRIGHTER GALACTIC HELIUM STARS BETWEEN GALACTIC LONGITUDES 150° AND 216°[1]—*Continued*

By J. C. KAPTEYN[2]

13. RANGE IN DISTANCE OR PARALLAX AS A FUNCTION OF λ AND v

The principle of the method followed is in the main that given by Eddington,[3] but the divergence from his treatment is sufficiently important to make a full explanation necessary. The parallaxes are derived from a comparison of the observed distribution of the v (given in Section 10) with the theoretical distribution obtained from the known distribution of u given in the preceding section.

We have (see *Mount Wilson Contribution* No. 82, p. 35)

$$v=0.212\,\pi(V\sin\lambda+u), \tag{50}$$

v and π being expressed in seconds of arc, V and u in km per second. Now, according to the preceding section, u is distributed about the central value $u=0$ in a normal error-curve having a probable error r_u. Therefore, if we consider only stars for which π and λ are practically constant, v will be distributed according to a normal curve about the central value

$$v=0.212\,\pi V\sin\lambda \tag{51}$$

with a probable error

$$r_v=\pm 0.212\,\pi r_u. \tag{52}$$

This, however, supposes that we disregard the observational errors. In the present case, where the proper motions are so very small, this is by no means permissible. But as the observational errors presumably are also distributed in a normal curve, the

[1] *Contributions from the Mount Wilson Solar Observatory*, No. 147.

[2] Research Associate of the Carnegie Institution of Washington, Mount Wilson Solar Observatory.

[3] *Stellar Movements and the Structure of the Universe*, p. 218.

observed values of v will have a similar distribution about the central value (51), only, instead of (52) we shall have to use

$$r_v = \pm\sqrt{(0.212\,\pi r_u)^2 + r^2} \tag{53}$$

where r is the probable observational error of a component of the proper motion in Boss's catalogue. I have used

$$r = 0''.0056 \tag{54}$$

which is the mean of the probable errors of the motions in α and δ, as given for all our stars in Boss's catalogue. Hence, finally,

$$r_v = \pm\sqrt{(0.212\,\pi r_u)^2 + 0.0056^2}. \tag{55}$$

The computations were first carried through with the first value of (49), namely,

$$r_u = \pm 2.5. \tag{56}$$

Table XVI gives the theoretical distribution for v derived in accordance with (51) and (55) for $\lambda = 150°$, $V = 20.0$ km,[1] on the supposition that the constant parallax has, successively, the values 0".0015,

TABLE XVI

DISTRIBUTION OF v ($\lambda = 150°$, $r_u = \pm 2.5$)

v	π							
	0".0015	0".0045	0".0075	0".0105	0".0135	0".0165	0".0195	0".0225
+0".0550	0.000	0.000	0.000	0.003	0.025	0.097	0.216	0.355
.0500	.000	.000	.000	.006	.031	.069	.092	.098
.0450	.000	.000	.002	.017	.056	.092	.108	.102
.0400	.000	.000	.007	.039	.087	.114	.114	.099
.0350	.000	.002	.021	.074	.118	.128	.111	.088
.0300	.001	.010	.053	.117	.142	.128	.101	.075
.0250	.004	.031	.102	.152	.146	.114	.084	.061
.0200	.017	.079	.159	.170	.134	.092	.064	.044
.0150	.058	.150	.191	.155	.105	.069	.045	.031
.0100	.129	.206	.183	.120	.072	.045	.029	.020
+0.0050	.206	.213	.140	.077	.044	.027	.017	.013
0.0000	.234	.163	.083	.041	.023	.014	.010	.007
−0.0050	.186	.092	.039	.019	.011	.006	.005	.004
−0.0100	.107	.039	.015	.007	.004	.004	.003	.002
	0.058	0.015	0.005	0.003	0.002	0.001	0.001	0.001

[1] In accordance with the definitive elements (33).

0″.0045, 0″.0075, 0″.0105, ... 0″.0225.[1] As a matter of fact, the computation was made for all the following values of π: 0″.0005, 0″.0015, 0″.0025, ... 0″.0235, and the totals in Table XVIII are the values obtained by this more elaborate computation. Indeed it is only in order to avoid typographical difficulties that the abridgment has been made. The abridged calculation is hardly less accurate, however, as will be seen if, in Table XVIII, the totals are formed for each line. Multiplied by 3, these differ but little from the totals obtained by the more elaborate computation.

Fig. 1 gives a comparison of the observed (see Table XI for $\lambda=150$) and theoretical distribution of v for $\pi=0''.0077$, which,

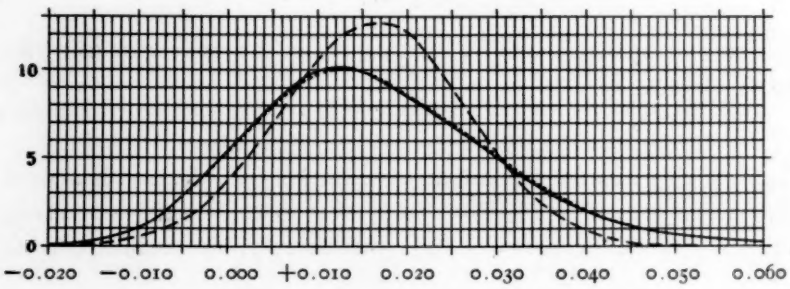

FIG. 1.—Distribution of the values of v for $\lambda=150°$

———— Observation
— — — — Theory, constant parallax, $\pi=0''.0081$
..... Definitive theory, interpolation from Tables XI and XXII

according to Table IX, is the mean parallax of the stars at $\lambda=150°$. It shows clearly that the observations are not represented at all. We conclude that the assumption of a constant π for all the stars must be abandoned. The deficiency of the theoretical curve in large values of v can evidently be remedied only by the addition of stars having larger parallaxes, while the deficiency in small values of v calls for stars of smaller parallax. Let us suppose, therefore, that there are (in fractions of the whole) n_5 stars of

[1] Stated as a formula, the tabulated quantity is

$$\text{Prob. } v_1 \text{ to } v_2 = \frac{1}{\sqrt{3.141}} \int_{H(v_1-0.212\,\pi V \sin\lambda)}^{H(v_2-0.212\,\pi V \sin\lambda)} e^{-z^2} dz$$

where $H=0.4769/r_v$ and r_v has the value given by (55).

parallax 0".0005; n_{15} stars of parallax 0".0015, etc. We must find n for different values of the parallax, that is, we must determine $n=\phi(\pi)$ in such a way that the mixture of all these stars gives a distribution for v agreeing with the observed distribution.[1]

Thus we might imagine each column of Table XVI to be multiplied by an indeterminate factor and then find the numerical values of these factors by equating the sums of the several horizontal lines to the corresponding numbers of the observed distribution. For instance, the sixth line of Table XVI, compared with the observed number from Table XI, would give:

$$0.001\, n_{15}+0.010\, n_{45}+0.053\, n_{75}+\ \ldots\ +0.075\, n_{225}=\frac{4.0}{66.0}=0.061.$$

We could thus determine all the constants, provided they were not more numerous than the equations of condition; but the solution would be neither complete—on account of the restricted number of constants—nor satisfactory. There are conditions which certainly, or at least very probably, must be satisfied if the solution is to be acceptable. As such we may specify:

$$\left.\begin{array}{l}\phi(0)=0 \qquad \phi(\infty)=0\\ \phi(\pi) \text{ shall have but one maximum and vary continuously;}\\ \bar{\pi}=\text{known value}^2\end{array}\right\} \qquad \text{(B)}$$

I therefore proceeded somewhat differently. Starting from a first approximation[3] for $\phi(\pi)$, which satisfies the conditions (B), I computed the theoretical distribution. The divergences of this from

[1] Put in the form of an integral equation: if $\psi(v)\delta v$ represent the fraction of the stars having their v between v and $v+\delta v$, we have

$$\psi(v)=\frac{1}{\sqrt{3.1416}}\int_0^\infty H\phi(\pi)e^{-H^2(v-0.212\,\pi\, V\sin\lambda)^2}\delta\pi$$

where

$$H=\frac{0.4769\ldots.}{\sqrt{(0.212\,\pi r_M)^2+r^2}}.$$ In this equation: given $\psi(v)$, required $\phi(\pi)$.

[2] This condition might have been omitted. The method will itself furnish the value of $\bar{\pi}$. I have preferred, however, to use the values already found.

[3] It is not of much importance what form we choose as a first approximation for $\phi(\pi)$. Still it will shorten matters not a little if it is not too far in error, and it is not difficult to find a fairly plausible beginning. Let $\psi(M)\delta M$ be the fraction of all the stars having absolute magnitude between M and $M+\delta M$; $D(\pi)$ = star density =

the observed distribution were then diminished by a local correction of $\phi(\pi)$, in which I took pains to keep (B) always satisfied. The process requires some care; but I was not long in obtaining a solution,[1] which, if it is not the best that can be obtained, meets the requirements in a very satisfactory manner. The solutions for $\lambda = 154°$ and $\lambda = 157°.5$ were carried through in exactly the same manner. The results for $\phi(\pi)$ are in Table XVII. The computation of the theoretical distribution for v with the help of these values $\phi(\pi)$ is shown for $\lambda = 150°$ in Table XVIII. The second column is obtained by multiplying the second column of Table XVI by 0.027, i.e. (see Table XVII), by the value of $\phi(\pi)$ for $\pi = 0''.0015$, and similarly for the other columns. These values of $\phi(\pi)$ are entered in the last line of the heading of Table XVIII. The theoretical distribution of v is given in the column headed Σ, which is the sum of all the numbers (for all the parallaxes $0''.0005$, $0''.0015$, $0''.0025$ . . . .) in the several horizontal lines. The last two

---

total number of stars per unit of volume at parallax π; m = apparent magnitude. Then it is easily found that

$$\phi(\pi) = \frac{A}{\pi^4} D(\pi) \psi(m + 5 + 5 \log \pi). \qquad (a)$$

It seems probable that we shall obtain a fair approximation if we assume the star-density to be constant, and if for $\psi(M)$ we use

$$\psi(M) = \frac{h}{\sqrt{\pi}} e^{-h^2(M-K)^2}$$

which was found for all the stars (see *Groningen Publication*, No. 11), and if finally for m we adopt the mean apparent magnitude of the Boss stars. We thus find

$$\phi(\pi) = \frac{B}{\pi^4} e^{-\lambda^2(G + 5 \log \pi)^2} \qquad (b)$$

in which there are two relations between the three constants expressing

$$\int_0^\infty \phi(\pi) \delta\pi = 1; \qquad \int_0^\infty \pi \phi(\pi) \delta\pi = \bar{\pi} = \text{known quantity.}$$

The third constant is easily obtained from the observed distribution for v by successive approximations. For stars of every spectrum considered together, *Groningen Publications*, No. 11, p. 11, gives in each column of Table II the $\phi(\pi)$ for a determined value of the apparent magnitude m. I find that these are in fact represented by formula (*b*) with surprising accuracy.

[1] It turns out that rather different solutions for π represent the distribution of v with almost equal precision; in other words, the solution for $\phi(\pi)$ is necessarily inaccurate. This does not mean that the determination of $\bar{\pi}$ as a function of v and λ, although based on $\phi(\pi)$, is not fairly reliable. In fact it is little affected by small changes in $\phi(\pi)$.

columns will be explained presently. Repeating the computation for $\lambda = 154°$ and $\lambda = 157°.5$, and multiplying the frequencies into the totals of stars observed, we have, for comparison with the observed numbers, the theoretical values in Table XI. The agreement, I think, may be considered satisfactory. For $\lambda = 150°$ the theoretical distribution is shown by the dotted line in Fig. 1.

TABLE XVII

$\phi(\pi)$ FOR $r_u = \pm 2.5$

π	λ			Mean
	150°	154°	157°.5	
0".0005.....	0.015	0.011	0.052	0.026
.0015.....	.027	.026	.087	.047
.0025.....	.044	.043	.087	.058
.0035.....	.066	.062	.079	.069
.0045.....	.084	.075	.071	.077
.0055.....	.093	.084	.063	.080
.0065.....	.095	.091	.056	.081
.0075.....	.096	.095	.053	.081
.0085.....	.093	.094	.050	.079
.0095.....	.087	.090	.047	.075
.0105.....	.078	.082	.044	.068
.0115.....	.066	.072	.042	.060
.0125.....	.054	.059	.040	.051
.0135.....	.041	.045	.037	.041
.0145.....	.028	.032	.033	.031
.0155.....	.019	.020	.031	.023
.0165.....	.010	.010	.028	.016
.0175.....	.003	.005	.024	.011
.0185.....	0.001	.003	.021	.008
.0195.....	.....	0.001	.018	.006
.0205.....	.....	.....	.015	.005
.0215.....	.....	.....	.011	.004
.0225.....	.....	.....	.008	.003
0.0235.....	.....	.....	0.003	0.001
$\bar{\pi}$	0".0079	0".0083	0".0083	.....

The values of $\phi(\pi)$ having thus been found, the derivation of the average parallax of all the stars having the same v and λ offers no difficulty.[1] Take for instance $\lambda = 150°$. From Table XVIII we

[1] We have:

$$\bar{\pi}_{v,\lambda} = \frac{\int_0^\infty H\pi\phi(\pi)e^{-H^2(v-0.212\pi V\sin\lambda)^2}\delta\pi}{\int_0^\infty H\ \phi(\pi)e^{-H^2(v-0.212\pi V\sin\lambda)^2}\delta\pi}$$

where

$$H = \frac{0.4769\ldots.}{\sqrt{(0.212\pi r_u)^2 + r^2}},\quad r = 0".0056.$$

find for the interval $v = 0''.0350$ to $0''.0300$, that is, practically for $v = 0''.0325$,

$$\bar{\pi} = \frac{0''.0045 \times 0.0008 + 0''.0075 \times 0.0051 + \ldots + 0''.0165 \times 0.0013}{0.0008 + \quad 0.0051 + \ldots \quad + 0.0013} = 0''.0116.$$

The more complete computation, carried through with three times the number of parallaxes, furnishes the more accurate value $\bar{\pi} = 0''.0107$ entered in the last column of Table XVIII. All other cases were treated in the same way.

TABLE XVIII

THEORETICAL DISTRIBUTION OF v TOGETHER WITH VALUES OF $\bar{\pi}$
(All Numbers in Units of the Fourth Decimal)

v	π AND $\phi(\pi)$							Σ	$\Sigma(\pi + \text{tab.})$	$\bar{\pi}$
	15	45	75	105	135	165	195			
	270	840	960	780	410	100	0			
+550..	0	0	0	2	10	10	0	65	0.95	146
500..	0	0	0	5	13	7	0	74	1.01	136
450..	0	0	2	13	23	9	0	140	1.81	129
400..	0	0	7	30	36	11	0	250	3.08	123
350..	0	2	20	58	48	13	0	420	4.85	115
300..	0	8	51	91	58	13	0	659	7.05	107
250..	1	26	98	119	60	11	0	943	9.27	98
200..	5	66	153	133	55	9	0	1,253	11.14	89
150..	16	126	183	121	43	7	0	1,484	11.79	79
100..	35	173	176	94	30	5	0	1,523	10.76	71
+ 50..	56	179	134	60	18	3	0	1,335	8.36	63
0..	63	137	80	32	9	1	0	953	5.27	55
− 50..	50	77	37	15	5	1	0	548	2.82	51
−100..	29	33	14	5	2	0	0	244	1.11	46
	16	13	5	2	1	0	0	110	0.46	42
Totals	271	840	960	780	411	100	0	10,001	79.73	79

Table XIX gives the complete results for $\lambda = 150°$, $\lambda = 154°$, and $\lambda = 157°.5$. For convenience I have added the values for $v = -0''.0125$, as found by extrapolation.

The argument of this table is the *observed* value of v, which of course is the argument needed for the computation of the parallaxes. For certain purposes, however, another table is desirable with the *true* value of v (freed of observational errors) as argument. This for instance would be useful for stars whose proper motions

are known with extreme precision. To calculate it we have only to use formula (52) instead of (53).[1] The results are given in Table XX.

TABLE XIX

$\pi = F(v, \lambda)$ ($r_\pi = \pm 2.5$) OBSERVED VALUES OF v

v obs.	λ		
	150°	154°	157°.5
+0".0525.....	0".0135	0".0142	0".0180
.0475.....	.0129	.0134	.0170
.0425.....	.0122	.0127	.0160
.0375.....	.0115	.0119	.0150
.0325.....	.0107	.0112	.0139
.0275.....	.0098	.0103	.0124
.0225.....	.0089	.0094	.0107
.0175.....	.0079	.0085	.0090
.0125.....	.0071	.0076	.0073
.0075.....	.0063	.0068	.0061
+0.0025.....	.0055	.0062	.0052
−0.0025.....	.0051	.0057	.0045
0.0075.....	.0046	.0053	.0042
−0.0125.....	0.0042	0.0050	0.0042

TABLE XX

$\bar{\pi} = F(v, \lambda)$ ($r = \pm 2.5$) TRUE VALUES OF v

True v	λ		
	150°	154°	157°.5
+0".070.....	0".0171	0".0187	0".0199
.060.....	.0159	.0175	.0195
.050.....	.0146	.0161	.0187
.040.....	.0132	.0142	.0173
.030.....	.0115	.0121	.0151
.020.....	.0092	.0099	.0119
.010.....	.0061	.0069	.0076
+0.005.....	.0043	.0058	.0046
0.000.....	.0052	.0055	.0034
−0.005.....	.0104	.0083	.0125
−0.010.....	0.0136	.00124	0.0154

The divergences of Table XX from Table XIX are in a direction easy to foresee. The last shows the curious fact that, as v decreases beyond a certain value, $\bar{\pi}$ begins to increase, and soon

[1] The substitution in the denominator of H of the value (52) for that of (53) is the only change in the formulae in the footnotes.

increases at a rapid rate. This, too, is in accordance with what might have been expected, as is perhaps best seen by comparing the mean parallax of stars (all at the same distance λ from the vertex) having $v = 0''.000$ with that of stars having a definite negative value of v. For stars at all distances, the probability of the value $v = 0''.000$ is the same, for it is simply the probability that the linear velocity component u is equal, but opposite in sign, to the stream-velocity component $V \sin \lambda$. In order that v shall be negative, the linear velocity component u must be greater than (and opposite to) the stream component $V \sin \lambda$. In addition, the more distant stars, in order to show the same negative value of v, must have the greater negative value of u. A greater negative value of u, however, means a less probable one. Consequently, whereas near and distant stars contribute equally in the production of $v = 0''.000$, the nearer stars predominate in the formation of a negative value of v. Therefore, the mean parallax will be greater in the latter case.[1] This assumes that there are no observational errors. That Table XIX, in which these errors are taken into account, must show quite another behavior is clear. Indeed it is easily seen that, without such errors, practically no negative values of v would occur, at least for stars at some distance from the vertices.

We now return to the parallaxes obtained by taking the observational errors into account. In Section 17 (*a*) evidence is given which indicates that it would have been preferable to have adopted the *same* values of $\phi(\pi)$ for all values of λ. I accordingly repeated the computation of the theoretical distribution of v and of the parallaxes, using in each case the mean values of $\phi(\pi)$ in the last column of Table XVII. These are direct means, notwithstanding the fact that the values for $\lambda = 154°$ depend partly on stars also

[1] In order to show more in detail the way in which $\bar{\pi}$ varies with v, I computed the following table on the supposition that half the stars have $\pi = 0''.0000$, the other half $\pi = 0''.0100$, and further that $V = 20.0$ km, $\lambda = 150°$, $r_u = \pm 2.5$ km.

v	$\bar{\pi}$	v	$\bar{\pi}$
$+0''.030$	$0''.0100$	$+0.005$	0.0073
0.025	0.0099	0.000	0.0080
0.020	0.0091	-0.005	0.0092
0.015	0.0078	-0.010	0.0099
$+0.010$	0.0072		

used for $\lambda = 150°$ and $\lambda = 157°.5$. The weight of the determination for $\lambda = 150°$ is undoubtedly considerably greater than that for $\lambda = 157°.5$. As the values for $\lambda = 154°$ are practically identical with those for $\lambda = 150°$, the manner of forming the means is sensibly equivalent to giving double weight to $\lambda = 150°$, which cannot be far wrong. Since the differences between the observed and the theoretical distribution were increased in every case by this procedure, the indication alluded to is not confirmed by the observed distribution of v. The safest course seems to be to adopt the arithmetic means of the parallaxes obtained on the two hypotheses, (*a*), $\phi(\pi)$ different for λ different; (*b*), $\phi(\pi)$ constant for all λ.

Having thus obtained the definitive parallaxes (for $r_u = \pm 2.5$) for $\lambda = 150°$, 154°, and 157°.5, we consider less frequently occurring values of λ. For $\lambda = 140°$, 145°, 165°, 180° I adopted the mean $\phi(\pi)$ in Table XVII and derived Table XXI, from which the individual parallaxes for $r_u = \pm 2.5$ were obtained by simple interpolation.

TABLE XXI

$\pi_{2.5} = \phi(v, \lambda)$ ADOPTED, FOR $r_u = \pm 2.5$ KM

v	λ							
	140°	145°	150°	154°	157°.5	160°	165°	180°
+0".0525	0".0143	0".0144	0".0145	0".0151	0".0172	0".0171	0".0170	.....
.0475	.0133	.0135	.0137	.0142	.0162	.0161	.0160	.....
.0425	.0124	.0126	.0128	.0133	.0151	.0151	.0150	.....
.0375	.0114	.0116	.0119	.0123	.0141	.0140	.0139	.....
.0325	.0104	.0107	.0110	.0115	.0130	.0129	.0127	0".0139
.0275	.0093	.0097	.0100	.0105	.0117	.0116	.0115	.0123
.0225	.0082	.0086	.0090	.0094	.0103	.0103	.0103	.0108
.0175	.0071	.0075	.0079	.0085	.0089	.0090	.0092	.0094
.0125	.0060	.0065	.0070	.0075	.0074	.0077	.0083	.0084
.0075	.0051	.0056	.0061	.0066	.0064	.0068	.0075	.0077
+0.0025	.0043	.0048	.0053	.0060	.0056	.0060	.0069	.0074
−0.0025	0.0035	.0042	.0048	.0054	.0050	.0055	.0065	.0074
0.0075	.....	0.0038	.0043	.0049	.0047	.0053	.0065	.0077
0.0125	.....	.....	0.0038	0.0046	0.0046	0.0052	0.0065	.0084
−0.0175	.....	.....	.....	.....	.....	.....	.....	0.0094

14. CASE $r_u = 0.0$

According to Section 12, Remark 2, the calculations have now to be repeated for the value $r_u = 0.0$. Tables XXII, XXIII, and XXIV, corresponding to Tables XI, XVII, and XXI, unchanged

but for the value of r_u, embody the results. Comparison of Tables XI and XXII shows that the observed distribution of v is about equally well represented in the two cases. The values of $\phi(\pi)$ in Tables XVII and XXIII show large differences. The remarks in the footnotes of Section 13 also hold here.

TABLE XXII*

DISTRIBUTION OF v (THEORETICAL NUMBERS ARE FOR $r_u=0.0$)

v	$\bar{\lambda}=150°$			$\bar{\lambda}=154°$			$\bar{\lambda}=157°.5$		
	Obs.	Theor.	O−C	Obs.	Theor.	O−C	Obs.	Theor.	O−C
+0".055	0.5	0.3	+0.2	0.3	0.2	+0.1	1.1	0.1	+1.0
.050	0.5	0.5	0.0	0.5	0.3	+0.2	0.7	0.5	+0.2
.045	0.9	0.9	0.0	0.8	0.5	+0.3	1.1	0.8	+0.3
.040	1.6	1.8	−0.2	1.6	1.5	+0.1	1.6	1.9	−0.3
.035	2.6	2.9	−0.3	2.7	2.7	0.0	2.5	3.0	−0.5
.030	4.0	4.6	−0.6	4.4	4.4	0.0	3.6	4.4	−0.8
.025	6.1	6.4	−0.3	6.5	6.4	+0.1	5.1	5.6	−0.5
.020	7.9	8.1	−0.2	8.4	8.4	0.0	6.8	6.9	−0.1
.015	9.2	9.4	−0.2	9.9	10.1	−0.2	8.6	8.4	+0.2
.010	10.2	9.6	+0.6	10.2	10.5	−0.3	10.0	10.0	0.0
+0.005	9.2	8.6	+0.6	9.0	9.4	−0.4	10.5	10.9	−0.4
0.000	6.7	6.4	+0.3	6.7	7.0	−0.3	9.7	10.0	−0.3
−0.005	4.1	3.8	+0.3	4.2	4.2	0.0	7.3	7.2	+0.1
0.010	1.8	1.8	0.0	2.1	1.9	+0.2	4.5	3.8	+0.7
0.015	0.6	0.8	−0.1	0.8	0.9	+0.2	1.9	1.7	+0.2
−0.020	0.1			0.3			0.5	0.4	+0.1
Totals	66.0	66.0	.....	68.4	68.4	.....	75.5	75.6	.....

* Same as Table XI except that $r_u=0.0$ instead of ± 2.5.

The parallaxes obtained by interpolation from Table XXI ($r_u=\pm 2.5$) appear in Table XXXIX under the head $\pi_{2.5}$; those

from Table XXIV ($r_u = 0.0$), under the head $\pi_{0.0}$. Finally the adopted π, corresponding to the adopted value $r_u = \pm 1.67$ (47), was obtained from

$$\pi = \frac{2\,\pi_{2.5} + \pi_{0.0}}{3}. \qquad (57)$$

TABLE XXIII*

$\phi(\pi)$ FOR $r_u = 0.0$

π	λ			Mean
	150°	154°	157°.5	
0".0005.....	0".040	0".032	0".119	0".064
.0015.....	.066	.058	.115	.080
.0025.....	.074	.066	.094	.078
.0035.....	.077	.069	.072	.073
.0045.....	.077	.073	.054	.068
.0055.....	.075	.075	.043	.064
.0065.....	.072	.072	.039	.061
.0075.....	.070	.070	.036	.059
.0085.....	.067	.067	.035	.056
.0095.....	.063	.063	.034	.053
.0105.....	.058	.058	.033	.050
.0115.....	.052	.052	.032	.045
.0125.....	.046	.046	.030	.041
.0135.....	.039	.039	.028	.035
.0145.....	.032	.032	.027	.030
.0155.....	.023	.029	.026	.026
.0165.....	.018	.024	.025	.022
.0175.....	.013	.019	.024	.019
.0185.....	.010	.016	.023	.016
.0195.....	.009	.013	.022	.015
.0205.....	.007	.011	.020	.013
.0215.....	.006	.008	.019	.011
.0225.....	.004	.006	.016	.009
.0235.....	0.002	0.002	.014	.006
.0245.....	.....	.....	.010	.003
.0255.....	.....	.....	.007	.002
0.0265.....	.....	.....	0.003	0.001
$\bar{\pi}$	0".0079	0".0085	0".0083	

* Same as Table XVII except $r_u = 0.0$ instead of ± 2.5.

For Boss 1517, for which v is far beyond the limits of the tables, see the footnote to Table XXXIX. For one or two stars there has been a slight extrapolation. For all stars outside the Nebula-group within 15° of the antivertex the values of π have been marked uncertain (:).

TABLE XXIV*

$\pi_{0.0}=\phi(v, \lambda)$ ADOPTED, FOR $r_u=0.0$

v	λ							
	140°	145°	150°	154°	157°.5	160°	165°	180°
+0".0525	0".0176	0".0186	0".0197	0".0203	0".0224	0".0226	0".0226	0".0081
.0475	.0160	.0171	.0182	.0191	.0212	0214	.0216	.0081
.0425	.0142	.0153	.0165	.0177	.0199	.0200	.0202	.0081
.0375	.0126	.0136	.0148	.0162	.0183	.0184	.0186	.0081
.0325	.0109	.0119	.0130	.0143	.0165	.0166	.0169	.0081
.0275	.0093	.0102	.0111	.0122	.0142	.0145	.0151	.0081
.0225	.0077	.0085	.0093	.0102	.0116	.0120	.0129	.0081
.0175	.0062	.0069	.0076	.0084	.0091	.0097	.0109	.0081
.0125	.0049	.0055	.0061	.0068	.0068	.0075	.0089	.0081
.0075	.0038	.0043	.0048	.0053	.0051	.0058	.0072	.0081
+0.0025	.0029	.0033	.0037	.0043	.0039	.0045	.0059	.0081
−0.0025	0.0026	0.0026	.0030	.0034	.0032	.0037	.0048	.0081
0.0075	.....	.....	.0025	.0027	.0028	.0034	.0045	.0081
−0.0125	.....	.....	0.0022	0.0023	0.0026	0.0032	0.0043	0.0081

* Same as Table XXI except $r_u=0.0$ instead of ± 2.5.

15. PROBABLE ERRORS OF THE PARALLAXES

We again consider separately the two cases $r_u=\pm 2.5$ km and $r_u=0.0$ km, beginning with $r_u=\pm 2.5$. The value of π adopted for any one star is really the mean $\bar{\pi}$ of the parallaxes of all stars having the same v and λ. Let π_t be the true parallax. It is easy to find the probable value of the deviations $\pi_t-\bar{\pi}$, if for the moment we assume that they follow the error-law. Take for instance a star for which $\lambda=150°$, $v=0".0325$. Table XVIII shows that for $r=\pm 2.5$ the adopted value of $\bar{\pi}$ ($=0".0107$) is the mean built up from

0.0008	stars having	$\pi=0".0045$,	therefore	$\pi_t-\bar{\pi}=-0".0062$
0.0051	" "	"$=0.0075$,	"	" $=-0.0032$
0.0091	" "	"$=0.0105$,	"	" $=-0.0002$
0.0058	" "	"$=0.0135$,	"	" $=+0.0028$
0.0013	" "	"$=0.0165$,	"	" $=+0.0058$

Hence, for a total of 0.0221 stars, the sum of the residuals, all taken positively, is

$$0.0008\times 0.0062+0.0051\times 0.0032+ \ldots . = 0.00004688.$$

Therefore

Average value of $\pi_t-\bar{\pi}=0".00212$
Probable value of $\pi_t-\bar{\pi}=0".00179$

The more refined computation, based on a table such as Table XVIII, but extended to all the parallaxes, 0".0005, 0".0015, 0".0025, ... gives

Probable value of $\pi_l - \bar{\pi} = \pm 0''.00191$.

With similar computations for other values of v and λ, and a repetition for $r = 0.0$, I find finally, interpolating for $r_u = \pm 1.67$, the results in Table XXV, which include the effect of observational errors in the proper motions. Roughly speaking, we may say that for $\lambda < 156°$

$$r_\pi = \pm 0''.0021. \tag{58}$$

For $\lambda > 156°$ the error rapidly increases.

TABLE XXV

PROBABLE ERRORS OF π

v	$\lambda = 150°$	$\lambda = 154°$	$\lambda = 157°.5$
+0".0325.....	0".0020	0".0021	0".0029
0.0225.....	0.0021	0.0022	0.0030
0.0125.....	0.0020	0.0022	0.0029
+0.0025.....	0.0019	0.0021	0.0025
−0.0075.....	.....	.....	0.0022

But these probable errors have been computed on the supposition that the deviations follow the error-law. This may answer for the larger values of v, but it gives only a rough idea of the deviations for the smaller values of v. Indeed it is seen from Table XVIII, in which each horizontal line gives the frequency of the various values of π for stars of the same λ and v, that the distribution becomes very skew indeed for the vanishing values of v. The skewness is still more marked for the higher values of λ, the tables for which are not given here.

It is evident that this must be so. None of the parallaxes is negative; therefore, as an example, in the case of $\lambda = 150°$, $v = 0''.0025$, where Table XVIII gives $\bar{\pi} = 0''.0055$, deviations below $-0''.0055$ are impossible, whereas no such impossibility exists for deviations greater than $+0''.0055$. In fact, according to Table XVIII, the number of such deviations is by no means negligible.

In *Groningen Publications*, No. 8, p. 21, I assumed—in a very similar case and for quite the same reason—that not π but $\log \pi$ is distributed according to a normal error-curve. Schwarzschild[1] demonstrated theoretically that this must really be so, or rather that it is consistent with what we know or accept in connection with the distribution of the stars in space.

Table XVIII and others similar to it for the case of vanishing observational errors seem to offer an opportunity for testing the validity of the assumption directly by what we know from observation, for these tables are really derived from the observed distribution of v. It may be sufficient to state that, on trial, I found the result somewhat inconclusive. For $\lambda = 157°.5$ the representation of the difficult cases by the logarithmic curve is satisfactory; for $\lambda = 150°$ it is not. More extensive data, less strongly influenced by observational errors, are required to make the test decisive.

16. APPLICATION OF THE METHOD TO THE VALUES OF τ

It seems natural to attempt to use the component τ of the proper motion as we have used v in what precedes. Since τ is independent of the stream-velocity, we avoid all the difficulties presented by this motion. We need not consider groups within narrow limits of λ; errors in its velocity V are of no consequence, etc.

We compare the observed distribution, which has been given in Table XII, with the theoretical distribution. To determine the latter we have, as before,

$$\tau = 0.212\pi t \tag{59}$$

(τ and π in seconds of arc; t in km per second). The distribution of t (Section 11) is practically according to an error-curve, the central value being $t = 0$, with a probable error

$$r_t = r_u = \pm 1.67 \text{ km} \tag{60}$$

Hence, the values of τ for stars of the same parallax will also be distributed according to the error-law. The central value will be zero, and the probable error $\pm 0.212\pi r_u$. The observational

[1] *Astronomische Nachrichten*, **190**, 361, 1912.

errors will not disturb the distribution in an error-curve; the only effect will be to increase the probable error to

$$R = \pm\sqrt{(0.212\pi r_u)^2 + r^2}$$

In Section 10 values of τ with probable errors exceeding $0''.0069$ were neglected. Consequently r is now found to be smaller than in the preceding section, namely,

$$r = \pm 0''.0047. \tag{61}$$

Hence

$$R = \pm\sqrt{(0.354\pi)^2 + (0''.0047)^2}\,. \tag{62}$$

In analogy with Section 13, we might now compute the distribution of τ for a series of consecutive values of π and then try to find the relative numbers of stars with these parallaxes necessary to produce a distribution of τ agreeing with that observed—in other words, find values of $n = \phi(\pi)$ which will harmonize theory with observation.[1] But it appears that such an attempt is doomed to failure, for, whether we assume all the stars to have the same parallax, or whether we assume π to be distributed according to $\phi(\pi)$ in Tables XVII and XXIII, we find practically the same distribution for τ. The solution is so little sensitive to changes in $\phi(\pi)$ that the determination of this function in the manner proposed is hopeless.

For example, let I represent the distribution of τ resulting from the known distribution of t, when we adopt for all the stars the parallax $0''.0081$. The distribution of τ then follows an error-curve with a probable error found by (62) to be $R = \pm 0''.0055$. Similarly let II represent the distribution of τ, resulting when we assume for $\phi(\pi)$ the values found for $\lambda = 150°$ by interpolation between Tables XVII and XXIII. Finally let III represent the same distribution, when for $\phi(\pi)$ we adopt the result of interpolating between the

[1] This would be equivalent to finding the solution of the integral equation:

$$\text{Prob.}\,{}_{\tau}^{\tau+\delta t} = \frac{1}{3.14\ldots.}\,\delta\tau \int_0^\infty H\phi(\pi)\varepsilon^{-H\pi^2}\delta\pi$$

where

$$H = \frac{0.4769\ldots.}{\sqrt{(0.212\pi r_u)^2 + r^2}}\,.$$

The first member being given, we would have to find $\phi(\pi)$.

mean values of Tables XVII and XXIII. The method of obtaining II and III is so analogous to that followed for v in Section 13 that it need not be described. The results (total number of stars 72) are given in Table XXVI. They are so nearly identical that, whatever

TABLE XXVI

π	Frequencies			No. of Stars		
	I	II	III	I	II	III
$>\pm$ 0".025.....	0.002	0.000	0.006	0.1	0.0	0.4
$\pm$0".025 to $\pm$0.020.	.012	.014	.018	0.9	1.0	1.3
$\pm$0.020 " $\pm$0.015.	.052	.052	.056	3.7	3.7	4.0
$\pm$0.015 " $\pm$0.010.	.154	.152	.154	11.1	10.9	11.1
$\pm$0.010 " $\pm$0.005.	.320	.318	.314	23.0	22.9	22.6
$\pm$0.005 " $\pm$0.000.	0.460	0.464	0.454	33.1	33.4	32.7
Totals.....	1.000	1.000	1.002	71.9	71.9	72.1

the observed distribution, a choice between the three solutions is impossible. All three agree equally well or equally badly.[1] Actually the agreement is in no case perfect, for solution I (therefore also II and III) represents the distribution in an error-curve with probable error 0".0055, while the observed distribution is sen-

[1] In the case of data a hundred times richer, such differences as we find between the several solutions might lie well outside the limits of uncertainty of the observed numbers. But even then the solution for $\phi(\pi)$ would have to be considered illusory, at least in the present state of science. To show this clearly, let us assume our data to be so numerous that the frequencies could be considered reliable to the third decimal inclusive, and further, that the observed frequencies agreed absolutely with the theoretical frequencies of solution II. Would this prove that solution II for $\phi(\pi)$ would be the correct one? By no means, for we are far from certain that the linear motion t for the distribution of which we assumed the error-law, really follows this law with a precision of a few thousandths in the frequencies. Indeed we know that, at least for a mixture of stars of all spectra, there are very appreciable deviations from this law (see Schwarzschild, *Astronomische Nachrichten*, **190**, 376, 1912; Kapteyn and Adams, *Proc. Nat. Acad. Sci.*, **1**, 18, 1915). It even seems questionable whether the accidental observational errors in t follow the error-law with such nicety. The real frequency law of the linear velocities might therefore be such that if we reduce to angular motion on the assumption of the same parallax for all stars, and add the effect of the accidental errors of observation, we would find, instead of I, such a distribution as II or III or any other distribution diverging from I by quantities of the same order. In case we found exactly the distribution II we should of course conclude, instead of inferring that $\phi(\pi)$ had the values given above for II, that all the stars had the same parallax.

sibly in an error-curve with the probable error $\pm 0''.0061$ (Section 10). Were the difference well established it would have to be interpreted, either (*a*) as an error in the adopted mean parallax $0''.0081$, or (*b*) as an error in the adopted probable amount ± 1.67 km for t, or (*c*) as a combination of both (*a*) and (*b*). On the first hypothesis the mean parallax should be $0''.0110$; but the evidence for the change is almost negligible.

17. ADDITIONAL EVIDENCE AS TO THE VALUES OF THE PARALLAXES

The difficulty of the question and particularly the relative crudeness of the data seem to justify the accumulation of as many checks as can be obtained. I have therefore brought together whatever further evidence I have been able to find, not hesitating to include some which is only partly independent of what precedes.

a) *The range in parallax.*—Though the values of τ do not afford a useful determination of the parallaxes, they may serve indirectly by rendering possible the elimination of observational errors. For this purpose I will compare the components v and τ for stars at the same, or nearly the same, distance λ from the vertex. The values of v contain the component of the stream-motion (in angular measure) and the peculiar proper motion; τ contains only the latter. If all the stars had the same parallax, the angular stream-motion in v would be the same for all. As the peculiar motions are supposed to be random in direction, the common angular stream-motion would be the arithmetic mean $\bar{v}$ of v. Hence $v-\bar{v}$, as well as τ, would include only the peculiar motion. Both would have all sorts of values, but the average amounts (all taken positively) and similarly the probable amounts would be equal.

If, on the contrary, the parallaxes are unequal, the stream-motion involved in the v will not be the same for all stars. The quantities $v-\bar{v}$ will include the peculiar motions, and also an element dependent on the stream-motion, which will be the greater, the greater the diversity of parallaxes.

The average value of $v-\bar{v}$ (disregarding signs) and its probable value will now be greater than the corresponding values for τ. The difference will be a measure of the diversity of the parallaxes, that is, of the range in distance.

In the elaboration of this idea we must consider the observational errors. Since the component of the stream-motion at right angles to the line of sight for stars at the distance λ from the vertex is $V \sin \lambda$ km per second, or $0.212\ V \sin \lambda$ astronomical units per year, we shall have, if we call α and β the peculiar angular motions in the direction of v and τ respectively,

$$\left.\begin{aligned} v &= 0.212\ \pi V \sin \lambda + \alpha + \text{obs. err. in } v \\ \tau &= \qquad\qquad\quad \beta + \text{obs. err. in } \tau \end{aligned}\right\}. \tag{63}$$

For a great number of stars the arithmetic means of α, of β, and of the observational errors in v and τ will be zero. Hence

$$\left.\begin{aligned} v - \bar{v} &= 0.212\ V \sin \lambda\ (\pi - \bar{\pi}) + \alpha + \text{obs. err. in } v \\ \tau &= \qquad\qquad\quad \beta + \text{obs. err. in } \tau^1 \end{aligned}\right\}. \tag{64}$$

Squaring and forming the means for all available stars, we find, putting

$$\pi - \bar{\pi} = \Delta\pi \tag{65}$$

$$\left.\begin{aligned} \frac{1}{n}\Sigma(v-\bar{v})^2 &= (0.212\ V \sin \lambda)^2 \overline{(\Delta\pi)^2} + \frac{1}{n}\Sigma\alpha^2 + \frac{1}{n}\Sigma(\text{obs. err. in } v)^2 \\ &\quad + \text{double products} \\ \frac{1}{n}\Sigma\tau^2 &= \frac{1}{n}\Sigma\beta^2 + \frac{1}{n}\Sigma(\text{obs. err. in } \tau)^2 \\ &\quad + \text{double products} \end{aligned}\right\} \tag{66}$$

The double products will nearly disappear, because the factors are as frequently positive as negative. Further, $\Sigma\beta^2$ will approximately equal $\Sigma\alpha^2$. The terms depending on observational errors may differ slightly on account of possible differences of accuracy in the proper motions for α and δ; but as these differences are generally small, and greatly diminished in v and τ, we assume these terms also to be equal. Hence, if we write

$$S^2 = \frac{1}{n}\Sigma(v-\bar{v})^2 - \frac{1}{n}\Sigma\tau^2 \tag{67}$$

we obtain

$$(0.212\ V \sin \lambda)^2\ \overline{(\Delta\pi)^2} = S^2 \tag{68}$$

where S is a quantity given by the observations.

[1] It is slightly more correct to use $\tau - \bar{\tau}$ than τ. This refinement was indeed introduced; $\bar{\tau}$, however, is so small that the correction is without influence.

Applying this formula to narrow zones of λ, I find

λ	$\overline{(\Delta\pi)^2}$ =	No. Stars	Weight
145° to 149°	+0.000011	26	15
150 " 154	+ .000042	40	12
155 " 159	+ .000025	28	4
160 " 164	+0.000017	17	1

(69)

from which

$$\overline{(\Delta\pi)^2} = +0.000024 \pm 0.000005 \quad (111 \text{ stars}). \tag{70}$$

From the preceding section, for λ = 150° and λ = 157°.5, the two being practically independent,

λ = 150°.0	$\overline{(\Delta\pi)^2}$ = 0.0000135
λ = 157.5	0.0000337
Mean	0.000024

The agreement of the two mean values is beyond expectation. As $\overline{(\Delta\pi)^2}$ is preponderatingly dependent on the extreme parallaxes, both large and small, I see in this agreement a good test of the correctness of the total range in distance. On the other hand, we here find a difference for high and low λ which is not confirmed by (69). As this difference is due to the differences shown in Tables XVII and XXIII between the values $\phi(\pi)$ for different values of λ, we obtain an indication that it would have been preferable to adopt for the different zones of λ values for $\phi(\pi)$ which more nearly agree. This is the reason why I definitively adopted in the preceding section, for λ = 150°, 154°, and 157°.5, values of $\phi(\pi)$ intermediate between those found directly and the mean of all.

b) *Test of the moderate and large parallaxes.*—Since v is the angle subtended at the earth by the projection on the sphere of the stream-velocity increased by the component u (u counted positively in the sense of the stream-motion), it will, on the whole, be somewhat greater for the stars having positive values of u, and smaller for those having negative values of u. In other words, there must be a certain degree of correlation between v and u. The coefficient of this correlation ρ necessarily lies between zero and unity, $\rho = 0$ corresponding to an absence of correlation. For this case the average of the several u's corresponding to any given v would be

zero; for $\rho=1$, u is a pure function of v alone, and a value of v fully determines u, which I will call u_v. According to (50) the two extreme cases give

$$\rho=0; \qquad \bar{\pi}_0=\frac{v}{0.212\,V\sin\lambda} \qquad (71)$$

$$\rho=1; \qquad \bar{\pi}_1=\frac{v}{0.212\,(V\sin\lambda+u_v)}. \qquad (72)$$

As the truth lies between the two, we thus obtain two limits for $\bar{\pi}$.

The value of u_v for any value of v is easily obtained. It will be necessary, however, first to derive from Table XI, which gives the distribution of the observed v, one giving the distribution of the true v (free from observational error). Professor Eddington[1] has given an extremely elegant and convenient method, but the observed frequency-curves are here so nearly error-curves that an even more convenient method is available.

Thus, for instance, the observed frequency-curve of Table XI for $\lambda=150^\circ$ is represented approximately by an error-curve with the central value $0''.0162$ and the probable error $0''.00917$. As the probable error of observation is $\pm 0''.0056$ (54), the frequency-curve of the v, freed from observational error, will evidently be an error-curve with the same central value, $\nu=0''.0162$, and the probable error $\sqrt{0.00917^2-0.0056^2}=\pm 0''.00725$. Proceeding in the same way for the other values of λ we have:

	Central Value	P.E. Free from Obs. Error
$\lambda=150^\circ$	$0''.0162$	$\pm 0''.00725$
$\lambda=154$	0.0159	0.00705
$\lambda=157.5$	0.0135	± 0.00693

* This is only approximately true and assumes that (with the exception of a vanishingly small number of extreme values of u) u is small as compared with $V\sin\lambda$. For in this case we may write, neglecting squares of small quantities,

$$\pi=\frac{v}{0.212\,(V\sin\lambda+u)}=\frac{v}{0.212\,V\sin\lambda}\left(1-\frac{u}{V\sin\lambda}\right)$$

from which, because $\bar{u}=0$,

$$\bar{\pi}=\frac{v}{0.212\,V\sin\lambda}.$$

[1] *Monthly Notices*, **73**, 359, 1913.

With the aid of these numbers we find by the ordinary theory of the distribution of observational errors that the following fractions of all the stars entered in Table XXVII have values of v exceeding given limits.

TABLE XXVII

FREQUENCY OF v EXCEEDING CERTAIN LIMITS

v	$\lambda=150°$	$\lambda=154°$	$\lambda=157°.5$
$>0''.030$.....	$0''.100$	$0''.089$	$0''.054$
.025.....	.206	.192	.132
.020.....	.362	.347	.263
.015.....	.544	.535	.442
.010.....	.718	.714	.633
0.005.....	0.851	0.851	0.906

Now it is evident that if u is a function of v alone, as it is in the extreme case under consideration, it increases and diminishes with v, and the number of values of v and of u exceeding specified corresponding limits must be the same, and conversely. That is, the value u corresponding to any particular value of v will be that value of u which is as frequently exceeded as is the particular value of v. Therefore, to take an instance, since by Table XXVII, for $\lambda=150°$, v exceeds $0''.030$ in 0.100 of all the cases, and since in the theory of observational errors we find that an error exceeding $+1.90$ times the probable error (which for u is ± 1.67) occurs in just 0.100 of all the cases, therefore $u_{0''.030}=+1.90\times 1.67=+3.2$, and similarly in other cases, thus giving the results in Table XXVIII. With the aid of these data, (71) and (72) lead to Table XXIX.

TABLE XXVIII

u_v ($r_u=\pm 1.67$)

v	λ		
	150°	154°	157°.5
	km	km	km
$0''.030$.....	+3.2	+3.4	+4.0
.025.....	+2.0	+2.2	+2.8
.020.....	+0.9	+1.0	+1.6
.015.....	−0.3	−0.2	+0.4
.010.....	−1.4	−1.4	−0.85
0.005.....	−2.6	−2.6	−3.3

TABLE XXIX

LIMITS FOR π

v	$\lambda=150°$			$\lambda=154°$			$\lambda=157°.5$		
	$\bar{\pi}_0$	π	$\bar{\pi}_1$	$\bar{\pi}_0$	π	$\bar{\pi}_1$	$\bar{\pi}_0$	π	$\bar{\pi}_1$
0".030...	0".0141	0".0124	0".0107	0".0161	0".0134	0".0116	0".0185	0".0162	0".0121
.025...	.0118	.0109	.0098	.0134	.0119	.0108	.0154	.0142	.0113
.020...	.0094	.0093	.0087	.0107	.0102	.0097	.0124	.0120	.0103
.015...	.0071	.0076	.0073	.0081	.0084	.0082	.0093	.0097	.0088
.010...	.0047	.0056	.0055	.0054	.0064	.0064	.0062	.0071	.0069
0.005...	0.0024	0.0037	0.0032	0.0027	0.0048	0.0038	0.0031	0.0041	0.0054

With these limits we have to compare the values of $\bar{\pi}$ corresponding to the true values of v as argument. For $r_u=\pm 2.5$ km, $\bar{\pi}$ has been given in Table XX; for $r=0.0$ it is to be computed by

$$\bar{\pi}=\frac{v}{0.212\,V\sin\lambda}=\frac{v}{4.24\sin\lambda}.$$

Interpolating for $r_u=\pm 1.67$, we find the values in Table XXIX under the heading π. Our test is that π ought to be included between the limits $\bar{\pi}_0$ and $\bar{\pi}_1$.

In making the comparison we must consider that the values of u_v in Table XXVIII depend in the last instance on observations, and as such are liable to error. The limit $\bar{\pi}$, itself, is therefore liable to such error. Further, from (72) we see that for values of u_v not too far from $-V\sin\lambda$ any error in u_v appears in $\bar{\pi}$ enormously exaggerated. For this reason it is not surprising that for values of v as small as 0".015 or 0".010 the values of π in Table XXIX are sometimes slightly beyond the limit $\bar{\pi}_1$. For v still smaller, the present test of the parallaxes is, I think, of little value. Taking all this into consideration there seems to me every reason to be satisfied.

c) Test for the distant stars.—I will here use only the Bo–B5 stars and will consider that fifth of them for which v is smallest—on the whole, the more distant objects. They are

for $\lambda=145°$ to $154°$ those for which $v<+0".0048$
for $\lambda=155$ to 165 those for which $v<+0.0015$.

Stars still nearer to the antivertex will be disregarded, as will also the few stars for which $\lambda<145°$. It is my aim to find something

like a lower limit for the parallax π_d of these stars which for brevity will be called the great-distance stars.

For this purpose I will make use of the luminosity-curve of the Bo–B5 stars, for which a fairly reliable curve has been found in *Mount Wilson Contribution* No. 82. A formula was also derived (*op. cit.*, p. 64) by which the mean parallax can be found for a group of stars which is complete down to a specified magnitude m_o, when the average apparent magnitude $\overline{m}$ of the whole group has been observed. For $m_o = 5.81$, which is the Harvard magnitude limit to which Boss's catalogue is complete, the formula becomes

$$\overline{m} = -4.115 - 5 \log \pi - 0.691 \frac{e^{-P^2}}{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{P} e^{-z^2} \delta z} \qquad (73)$$

where[1]

$$P = +4.055 + 2.045 \log \pi. \qquad (74)$$

In order to eliminate to a great extent the errors of the adopted luminosity-curve, I will apply the formula, not only to the great-distance stars, but also, for comparison, to the entire group of Bo–B5 stars, adopting for the mean parallax of the latter $\overline{\pi} = 0''.0081$. The method thus becomes a more or less differential one.

If to the limit 5.81 mag.

$\overline{m}_1$ = average mag. of all the stars

$\overline{m}_2$ = average mag. of the great-distance stars

then (73) and (74) show that

if $\overline{m}_2 - \overline{m}_1 =$		$0^m.00$	then $\pi_d =$		$0''.0081$
"	"	$+0.10$	"	"	0.0069
"	"	$+0.20$	"	"	0.0057
"	"	$+0.30$	"	"	0.0046
"	"	$+0.40$	"	"	0.0036

From Table XXXIX I find that in reality

$\overline{m}_1 = 4.78 \pm 0.06$ (86 stars)

$\overline{m}_2 = 4.81 \pm 0.14$ (14 stars)

[1] The formula supposes that the range in π is not excessive. I feel confident that we may safely apply it in the present case. But, in case of doubt, the method which is fully explained in *Mount Wilson Contribution* No. 82, pp. 65–68, can be applied.

Therefore $m_2 - m_1 = +0.03 \pm 0.15$, and according to the preceding table

$$\pi_d = 0''.0077 \pm 0''.0018. \tag{75}$$

Since the average value of v for the stars of great distance is found to be $-0''.0016$, interpolation between Tables XXI and XXIV gives for the mean of $\lambda = 150°$, $154°$, and $157°.5$

$$\pi_d = 0''.0046. \tag{76}$$

The difference is considerable, but so also is the probable error in (75). In fact the difference is 1.7 times the probable error.

The conclusion from this test would seem to be that the number of stars is too small to lead to any well-founded result. The result as it stands would indicate that the adopted parallaxes of the distant stars are too low rather than too high. Any material change in this direction seems pretty well excluded, however, by the more reliable tests (*a*) and (*b*).

d) *Test by binary systems.*—Professor Hertzsprung courteously furnished me with the following hypothetical parallaxes, derived by a method which allows of a fairly good determination of parallax for any double star as soon as it shows sensible orbital motion. He will no doubt soon publish his method. In addition I quote the parallax of δ Orionis according to Stebbins.[1] For comparison the parallaxes obtained in the present paper are added.

	Hertzsprung and Stebbins	Kapteyn
ζ Orionis = Boss 1398..	$0''.0082$	$0''.0057$ neb.
σ " = " 1389..	0.0060	0.0057 neb.
15 Monocerotis = " 1706..	0.0089	0.0057
32 Orionis = " 1331..	0.0138	0.0156
8 " = " 1339..	0.0032	0.0057 neb.
Mean.....	0.0080	0.0077

18. THE NEBULA-GROUP. ITS PARALLAX

In this group the proper motions are so exceedingly small that the method for finding the stream-elements and the parallax used

[1] *Astrophysical Journal*, **42**, 145, 1915.

above fails altogether. Fortunately I had at my disposal for this region and its surroundings, namely,

$$\text{R.A. } 5^h0^m \text{ to } 6^h0^m, \quad \text{Dec. } -10° \text{ to } +20°, \tag{77}$$

the spectra for the A and B stars obtained by Miss Cannon for the *Revised Draper Catalogue*. In Table XXX the numbers of these spectra have been summarized. In order to be sure of including only the densest part of the Nebula-group, I narrowed the limits (2) of this group[1] to

$$\text{R.A. } 5^h20^m \text{ to } 5^h40^m, \quad \text{Dec. } -7° \text{ to } +2°, \tag{78}$$

an area of 45 square degrees.[1] For comparison I included the stars of region (77) well outside the Nebula-group. For these I adopted the limits:

$$\left.\begin{array}{ll} 5^h\ 0^m \text{ to } 5^h20^m & +\ 3° \text{ to } +20° \\ 5\ 20\ \text{“}\ 5\ 60 & +\ 5\ \text{“}\ +20 \\ 5\ 0\ \text{“}\ 5\ 10 & -10\ \text{“}\ +\ 3 \\ 5\ 50\ \text{“}\ 5\ 60 & -10\ \text{“}\ +\ 5 \end{array}\right\} \tag{79}$$

containing in all 305 square degrees. The space between these regions was not used in order to be independent of any uncertainty in the exact limits of the Nebula-group.

The last line but one in Table XXX gives the mean of the observed magnitudes. The last line gives the ratio of star density in nebula region to star density outside this region. We thus find that the nebula region is fully twelve times richer in Bo–B5 stars than are the surroundings. For later types this ratio rapidly diminishes. The Nebula-group thus contains both B and A stars, but, as compared with the surrounding sky, it is very much richer in spectra of the former class.

The striking fact in the numbers of Table XXX is that the magnitude of greatest frequency is different for the nebula and the region outside. This at once promises a determination of the ratio of the parallaxes of the two groups of stars. For the absolute magnitudes of stars which are practically at the same distance

[1] This narrowing of the limits is the reason why the number of bright B stars in the Nebula-group, as shown by Table XXX, is smaller than that in Table XXXIX. Moreover the Oe5 stars have been omitted from Table XXX.

TABLE XXX

NUMBER OF STARS IN "REVISED DRAPER CATALOGUE"

MAGNITUDE LIMITS	MEAN MAG.	B3		B5		B0–B5		B5–B9		B8–B9		A	
		Neb.	Outside	Neb.	Outside	Neb.	Outside	Neb.	Outside	Neb.	Outside	Neb.	Outside
9.05–9.55	9.3	0	0	0	0	0	0	12	7	12	7	48	87
8.55–9.05	8.8	0	0	2	0	2	0	52	31	50	31	43	207
8.05–8.55	8.3	3	0	3	3	6	3	26	45	23	42	21	99
7.55–8.05	7.8	3	3	3	2	6	6	18	31	15	29	6	35
7.05–7.55	7.3	3	0	1	0	4	0	5	9	4	9	2	13
6.55–7.05	6.8	1	1	2	2	3	3	4	11	2	9	4	8
6.05–6.55	6.3	4	3	4	....	11	4	7	4	3	4	0	13
5.55–6.05	5.8	4	1	....	....	6	3	....	7	....	7	2	2
5.05–5.55	5.3	0	1	....	....	3	2	....	3	....	3	....	8
4.55–5.05	4.8	2	2	....	....	3	2	....	1	....	1	....	0
4.05–4.55	4.3	....	1	....	....	0	3	....	....	....	....	....	1
3.55–4.05	3.8	....	....	....	....	1	0	....	....	....	....	....	0
3.05–3.55	3.3	....	....	....	....	0	0	....	....	....	....	....	0
2.55–3.05	2.8	....	....	....	....	0	0	....	....	....	....	....	1
2.05–2.55	2.3	....	....	....	....	2	0	....	....	....	....	....	....
1.55–2.05	1.8	....	....	....	....	1	1	....	....	....	....	....	....
1.05–1.55	1.3	....	....	....	....	....	....	....	....	....	....	....	....
0.55–1.05	0.8	....	....	....	....	....	....	....	....	....	....	....	....
0.05–0.55	0.3	....	....	....	....	....	....	....	1	....	1	....	....
Total no		20	12	15	7	48	27	124	150	109	143	126	474
Mean mags		6.79	6.17	7.50	7.73	6.42	6.06	....	....	....	....	....	....
Density Nebula / Density outside		11.3		14.5		12.0		5.6		5.2		1.83	

differ from the apparent magnitudes by a constant, $5+5\log\pi$.[1] Consequently the frequency-curve of the apparent magnitudes differs from the luminosity-curve (which is the frequency-curve of the absolute magnitudes) merely in that there is a displacement of $5+5\log\pi$ mags. between the two. For two groups with parallaxes π_1 and π_2 the frequency-curves of the apparent magnitudes will thus be the same, with the exception of the displacement

$$5\log\frac{\pi_2}{\pi_1} \text{ magnitudes.} \qquad (80)$$

Consider the region of the nebula and that defined by (79) lying outside the nebula. As to the former, the lateral dimensions being small, it is all but certain that the depth also will be small. For the outside stars the distances must vary to a much greater extent,

[1] See, for instance, *Groningen Publications*, No. 11, p. 12.

and the frequency-curve of the apparent magnitudes will here consist of the superposed frequency-curves for the different distances. If we suppose the curve of the absolute magnitudes to be an error-curve (see Section 27), the frequency-curve of the apparent magnitudes for stars at the average distance will be an error-curve with its maximum at a certain magnitude. For the stars of smaller parallax the curve will be shifted to the fainter magnitudes; for the stars with greater parallax, to the brighter magnitudes. As long as the range in distance for the great majority of the stars is moderate, as is here the case, the superposition of all these curves will still be practically an error-curve, with its maximum at nearly the same magnitude as for the stars of average parallax. Only the probable error of the curve will have increased.

This is just what is seen in Table XXX. Take first the Bo–B5 stars. Owing to the limited data the numbers run very irregularly. There is, however, the fortunate circumstance that for both regions the number of stars fainter than 9.0 is zero; for the outside stars there is none fainter than 8.5. I conclude that, in the latter case at least, we have before us the *complete* frequency-curve. In the nebula region this is less certain; still the number of stars fainter than 9.0 is certainly small. It seems reasonable to assume that the true number of stars fainter than 9.05 (adopted average magnitude 9.3) is between zero and six. Now, if the luminosity-curve is an error-curve, or any symmetrical curve, the lateral shift will be that of the center of gravity or of the arithmetic mean. The mean of the magnitudes for the outside stars is 6.06. For the Nebula-group it is 6.42 and 6.74, respectively, for the two limits adopted. The displacement therefore lies between 0.36 and 0.68 mags. As the whole difference corresponds to only 0".0010 in the parallax we may adopt

$$\text{Bo–B5 stars} \qquad \text{Displacement} = 0.52 \text{ mag.} \tag{81}$$

For the B8–B9 stars the matter is more difficult. But I think we may assume with some confidence that the numbers for magnitudes 9.0 and brighter are complete: for, (*a*) the whole number of stars for which Miss Cannon has determined the spectrum is 240,000,[1] which, according to *Groningen Publications*, No. 18, involves average

[1] *Vierteljahrsschrift der Astronomischen Gesellschaft*, **51**, 85, 1916.

completeness to magnitude 9.5 (Harvard Scale); (*b*) the B and A stars, being very white, must in general have been observed to a fainter limit (visually) than the others; (*c*) according to Table XXX so many stars fainter than 9.05 have been observed in the regions now under consideration that it is improbable that the brighter stars should be seriously incomplete.

On the contrary, the numbers between 9.05 and 9.55 in Table XXX are almost certainly incomplete. To be comparable with the rest they must be multiplied by a factor ϵ, greater than unity. For the nebula region, ϵ cannot well exceed 2, as is shown by the A stars, which at magnitude 9.3 must be near the maximum of their frequency-curve of apparent magnitude; the total number, therefore, from 9.05 to 9.55 can scarcely be more than double that between 8.55 and 9.05. If we admit this the uncertainty about the true value of ϵ is of little consequence, for in attempting to fit an error-curve to the observed number of the B8–B9 stars in the nebula region, we find the maximum at practically 8.8, both for $\epsilon=1$ and $\epsilon=2$.

For the outside region it seems better to ignore the stars fainter than 9.05 altogether. From the others it appears that the maximum cannot lie far from 8.30. We are thus led to a displacement of 0.50 mag. Another estimate of this quantity made in a somewhat different way gave 0.37 mag.[1] I finally adopt:

B8–B9 stars Displacement = 0.44 mag. (82)

The displacements (81) and (82) require a correction. In the case of types B0–B5, the stars within the limits (78) of the Nebula-group are twelve times more crowded than those in the surrounding regions. We conclude that a twelfth part of the B0–B5 stars within these limits do not belong to the group, but are merely seen projected on it. These can have no displacement relative to the outside stars, to which they really belong. For the others, it must

[1] In the countings of the Pickering-Cannon stars, there is, as in other investigations of the kind, an obvious excess of stars of magnitude 9.0, 8.5, 8.0, and a deficiency for magnitudes 9.1, 8.9, 8.1, 7.9. I also carried through a computation in which this error was corrected; as the corrections for the numbers of stars within the half-magnitude intervals of Table XXX are insignificant, I will not burden this paper with the computation.

be increased by an eleventh in order to correct the displacement (81). For similar reasons the displacement (82) must be increased by $\frac{1}{4.2}$ its amount.

Hence:

B0–B5 stars: corrected displacement = 0.57 mag. (weight 2)
B8–B9 stars: corrected displacement = 0.54 mag. (weight 3)

The weights are estimates; the weighted mean is

$$\text{Displacement} = 0.55 \text{ mag.} \tag{83}$$

and by (80)

$$\frac{\bar{\pi} \text{ Outside Stars}}{\bar{\pi} \text{ Nebula-group}} = 1.288 \tag{84}$$

It remains to find $\bar{\pi}$ for the stars of magnitude 7.8, which is the average of the Cannon stars. For the Boss stars, of average magnitude 4.7, $\pi = 0''.0081$. From the Table XXXIX I find[1]

Average Mag.	$\bar{\pi}$	No.
2.73	0".0075	7
4.16	0.0083	14
5.04	0.0086	43
5.81	0.0083	38

Apparently there is no change with the magnitude; but, as it is improbable that there should be no change at all, I adopt as a conservative estimate,

$$\bar{\pi} = 0''.0077 \ (\bar{m} = 7.8, \text{ outside stars}), \tag{85}$$

with the aid of which by (84)

$$\bar{\pi} \text{ (Nebula-group)} = 0''.0060. \tag{86}$$

I think that the error of the displacement (83) cannot well be greater than half its amount and that its probable amount will not exceed a fourth of the whole. The probable error of the difference of the average parallaxes of the two groups of stars here considered ought not, therefore, to exceed 0".0005 and, as the probable error of

[1] Preliminary values of the parallaxes were used. For the present purpose these are sufficient.

the average parallax of the outside Boss stars is, by (33), $\pm 0''.0007$, the probable error of (86) may be estimated at $\pm 0''.0009$. It is found in Section 26, however, from a discussion of the luminosity-curve of the A stars, that the value (86) is very probably somewhat high, and I finally adopted

$$\bar{\pi}\ (\text{Nebula-group}) = 0''.0054 \pm 0''.0009. \tag{87}$$

The lateral dimensions of the Nebula-group, according to Section 3, are about 7° in right ascension and 14° in declination. If, as a crude approximation, we suppose that the group is roughly spherical with a diameter of 10°.5, the parallaxes will range from $0''.0049$ to $0''.0060$. For that half of the stars for which the proper motions are smallest I assume the parallax to be $0''.0051$, and for the others, $0''.0057$; that is,

$$\left.\begin{array}{ll} \text{for } 100\,\mu \geqq 0''.8 & \pi = 0''.0057 \\ \text{for } 100\,\mu < 0''.8 & \pi = 0''.0051 \end{array}\right\} \tag{88}$$

As already mentioned, one-twelfth of the Bo–B5 stars within the limits (2) do not belong to the physical group. Within these limits there are in Table XXXIX 26 stars, practically all Bo–B5 stars. Two or three are, therefore, probably outside the group, but I have assumed that the four, Boss 1302, 1343, 1376, and 1382, the only ones having $\mu \geqq 0''.020$, are in this class. The last has a large negative value of v and perhaps ought to be excluded altogether. The motions of the others can be satisfactorily explained as the result of traces of stream-motion, peculiar motion, and observational error.

Five stars, Boss 1454, 1277, 1435, 1512, and 1297, having spectra Bo–B5, are within 5° of the adopted limits (2) of the Nebula-group. Their proper motions are nearly vanishing. It is doubtful whether they belong to the Nebula-group. As their parallaxes computed on the hypothesis that they are group stars differ in no case more than 30 per cent from those obtained on the assumption that they are outside stars, I adopt the means for the two suppositions.

19. TESTS FOR THE PARALLAX OF THE NEBULA-GROUP

a) *By the luminosity-curve of the Bo–B5 stars.*—From the third column of Table XXX, we have concluded that the maximum of

the frequency-curve for the Bo–B5 stars in the Nebula-group lies between

$$m = 6.42 \quad \text{and} \quad m = 6.74. \tag{89}$$

In *Mount Wilson Contribution* No. 82, p. 43, the absolute magnitude corresponding to this maximum was found to be

$$M = 0.885. \tag{90}$$

If, therefore, we adopt the mean of (89) we find from

$$M = m + 5 + 5 \log \pi, \tag{91}$$

$$\pi = 0''.0072. \tag{92}$$

I have not used this for the improvement of the parallax (87) because it is preferable to follow the inverse course, that is, use (87) for converting the apparent magnitudes of Table XXX into absolute magnitudes, which will be employed to improve the luminosity-curve.

b) *Binary systems.*—Among the binaries of Section 17, the three marked "neb." belong to the Nebula-group. In the mean they give $\pi = +0''.0058$, agreeing almost perfectly with the value $0''.0057$ derived from (87). That these binaries were also used in the test of the parallax of the outside stars is justified by the fact that our determination of the parallax of the Nebula-group is based on that of the outside stars.

20. MOTION OF THE NEBULA-GROUP

Considering only the best-observed stars ($100\,r < 0''.040$) within the limits (2) of the Nebula-group, we derive from 17 stars the averages:

$$\bar{\alpha} = 5^h25^m; \quad \bar{\delta} = -1°.4; \quad 100\,\bar{\tau} = +0''.03; \quad 100\,\bar{v} = +0''.02. \tag{93}$$

Including further all the stars of known radial velocity, we find from 13 stars:

$$\bar{\alpha} = 5^h26^m; \quad \bar{\delta} = -2°.2; \quad \bar{\rho}_{\text{corr.}} = \bar{\rho} - 4.3 = 18.46 \text{ km} \tag{94}$$

The value (93) of $\bar{v}$ being absolutely insensible, I conclude that the antivertex must be not far from the point

$$5^h25^m, \qquad -1°.4. \tag{95}$$

How far away can it be placed without conflicting with the observations? I think it can safely be said that, in an average of 17 well-observed stars, a value of $100\,\bar{v}$ as high as $0''.5$ is irreconcilable with (93). It seems worth while in this connection to remark that the 12 outside stars, for λ between 170° and 178°, give

$$100\,\bar{v} = +0''.46 \text{ from which } \bar{\pi} = 0''.0083. \tag{96}$$

Notwithstanding the extremely unfavorable circumstances we still find a parallax agreeing perfectly with the final result (33). Now, from (51) and (16), eliminating V,

$$\tan\lambda = \frac{v}{0.212\,\rho\pi}, \tag{97}$$

from which, if we put $\pi = 0''.0054$; $v = +0''.0050$; $\rho = +18.46$, we obtain

$$\lambda = 13°. \tag{98}$$

It is to be considered extremely probable, therefore, that the antivertex of the Nebula-group lies within 13° of (95). The antivertex of the outside stars is at

$$5^h44^m, \qquad -11° \tag{99}$$

which is 10°.7 from (95). The radial velocity of the outside stars at the point (94), which is 9°.9 from (99), is $20.0 \cos 9°.9 = 19.7$, which differs only 1.24 km from (94). It is very possible, therefore, that both the direction and the velocity of the Nebula-group are identical with those of the outside stars. At all events the difference must be relatively small.

[*To be concluded*]

ON THE CAUSE OF THE DISTANCE-VELOCITY EQUATION IN STELLAR MOTIONS

By C. D. PERRINE

A series of dependences of radial velocities upon the magnitudes, spectral classes, size of proper motions, and distances of the stars have been found by various investigators in recent years which have proved most puzzling. All of these relations have been shown with more or less certainty by various methods of discussion, in addition to the two-stream or preferential motion.

The finding of what is believed to be evidence of the action of cosmical matter as a factor in the spectral conditions of the stars has led me to consider the bearing of such a hypothesis on the motions of the stars. Before discussing this bearing, however, it is necessary to attempt to determine whether these relations are to distance alone or to spectral class and absolute magnitude also. I have investigated to some extent these apparent relations, particularly the dependence upon apparent brightness, and accepted the evidence of these relations until quite recently, when I found[1] that if the ellipsoidal motion was taken into account simultaneously practically all of the other peculiarities could be satisfied upon the hypothesis that the more distant stars, as inferred from their proper motions, were in general moving more slowly than the near stars.

Adams and Strömberg[2] find that when the apparent magnitudes of F, G, K, and M stars were reduced to absolute magnitudes, by means of parallaxes measured directly, spectroscopically, or derived through proper motions, there was shown a strong relation of the radial velocities to these absolute magnitudes and a consistent difference between the velocities of the F and G stars and the K and M stars. As the element of distance which exists in the apparent brightnesses is eliminated in the absolute magnitudes, the conclusion that the dependence is upon absolute magnitude and

[1] *Astrophysical Journal*, **46**, 266, 1917. [2] *Ibid.*, **45**, 293, 1917.

spectral class appears to conflict with my own, stated above, viz., that the dependence is really upon distance after the ellipsoidal or two-stream motion has been allowed for.

Adams and Strömberg did not take into account directly stream-motion but believe that such influence upon their results from absolute magnitude is slight.[1] In my own investigation the effect of stream-motion in the nearer stars and an eccentricity or skewness in the distant stars showed so prominently that I am still in doubt whether these effects may not be sensible unless special precautions are taken to eliminate them. Some indication of such a dependence when stream-motion is not taken into account is found in my own investigation, which otherwise shows no such dependence, and will be referred to later.

The strongest evidence for doubt as to there being a dependence of velocity upon absolute magnitude is found in Tables VIII and IX of my article[2] where the data are in a form to show at once any considerable effect of absolute magnitude. These tables are reproduced in the present article as I and II. In the stars of small μ (Table I) the average μ is not given for the different groups of magnitudes, but it will probably not differ very greatly, as the same superior limit was adopted for all. ρ_2 shows no dependence upon magnitude. ρ_1 shows for the group of brighter stars a decidedly smaller value than for the fainter groups, but in the face of the other groups with many more stars can scarcely have much weight.

Assuming that the distances are essentially the same for all of these groups, the grouping by apparent magnitude becomes a grouping according to *relative absolute magnitude* also.

In Table II the range of values of μ is considerable and is taken into account. If we assume that the distances for such groups as these are proportional to their proper motions, then the values of ρ_2' and ρ_1' become the values reduced to a common unit for the three classes of apparent brightness, and therefore equivalent to *relative absolute magnitudes*.

These groupings leave the effects of two factors undetermined, viz., spectral class and the effect of the solar motion when classified according to the ellipsoidal axis. The distribution of the

[1] *Loc. cit.*, p. 302. [2] *Astrophysical Journal*, 46, 278, 1917.

spectral types is probably sufficiently uniform, with the possible exception of the stars of 2.9 and brighter in Table I where there is probably an excess of stars of class B. In this connection it is to be noted that, in the investigation already referred to, no dependence upon spectral class was shown when streaming was directly allowed for. The effect of the neglected solar motion may be appreciable on the separate values of ρ but is not likely to be in the separate groups under either ρ_2 or ρ_1, which in the present case is the important consideration.

TABLE I*

CLASSIFICATION ACCORDING TO BRIGHTNESS

SMALL μ

	ρ_2	ρ_1	ρ_2/ρ_1
	km	km	
$2^{M}9$ and brighter...	(34) 13.5	(34) 6.2	2.18
3.0 to $5\frac{1}{2}$.....	(297) 13.6	(317) 11.8	1.15
$5\frac{1}{2}$ to $6\frac{1}{2}$.....	(179) 12.4	(169) 10.4	1.19

TABLE II*

CLASSIFICATION ACCORDING TO BRIGHTNESS

LARGE μ

	ρ_2		ρ_2'	ρ_1		ρ_1'	ρ_2/ρ_1	ρ_2'/ρ_1'
	km	μ	km	km	μ	km		
$2^{M}9$ and brighter.	(15) 16.6	0".42	39.5	(28) 13.8	0".62	23.1	1.28	1.71
	(13) 15.9	.29	54.8	(26) 13.6	.38	35.8	1.17	1.53
3.0 to $5\frac{1}{2}$.....	(96) 26.4	.38	69.3	(212) 17.6	.44	40.2	1.57	1.72
$5\frac{1}{2}$ to $6\frac{1}{2}$.....	(31) 45.2	.64	70.6	(49) 22.3	.56	39.8	2.03	1.77

* ρ_2 represents the major axis of the ellipsoid, the limits being 50° from either vertex. ρ_1 represents the minor axis, the limits being from 60° to 90° from the vertices. ρ_2' and ρ_1' are the corresponding quantities reduced to a common value of μ. ρ_2/ρ_1 and ρ_2'/ρ_1' are the prolatenesses of the respective ellipsoids.

All of the fainter stars and one of the groups of brighter stars, comprising 95 per cent of the whole number (1457), show no increase of velocity with decreasing brightness, nor upon the foregoing assumptions, with decreasing absolute magnitude. The brighter stars of large μ and one group of the brighter stars of small μ are smaller than fainter stars of their respective classifications. The small number of such stars, 5 per cent of the whole, make their evidence doubtful in the face of the other, which is very definitely to the contrary.

A rough recombination of this data to neutralize the classification by stream-motion, according to the number of stars and on the assumption that such a grouping gives velocities according to absolute magnitude, shows an increase of velocity amounting to 1 km per absolute magnitude between the first and second groups, but no appreciable increase between the second and third groups. This result, although not of great weight, shows that a classification of these stars in which the peculiarities of preferential motion have not been definitely allowed for may produce a dependence upon absolute magnitude.

Further examination of Adams and Strömberg's results shows an apparent relation of velocity to size of parallax in the group of largest parallax in both the F and G group and the K and M group, as well as indications of some effect of a small number of large velocities among the absolutely fainter stars. They state:[1] "The increase in the value of the radial velocity with the absolute magnitude is shown clearly for essentially all of the groups in Table I. It is most marked in the case of the stars of large parallax, since these show the largest range in absolute magnitude, and their parallaxes are most accurately determined." From this I understand that most confidence is placed in the evidence from the stars of larger parallax, a confidence which, I think, will be shared by all investigators.

Now it is precisely in these two groups of their Table IV where the evidence seems to be doubtful. In the first place there is a small but well-defined progression in the mean parallaxes of the groups *in the same direction as the progressions in velocity. If the values of* $\underline{V'}$ *are reduced to unity upon the assumption of a linear relation between* π *and* $\underline{V'}$ *the continuity of the progression in the F and G stars is greatly weakened and the progression in the K and M stars destroyed.* The results are given in Table III and Fig. 1.

The method of geometric means used by Adams and Strömberg avoids, as they point out, the undue effect of large individual values of velocity. Forty-two of such inclusions appear to have been used by them, of which 31 are of absolute magnitude 3.3 and fainter, or 36 of absolute magnitude 2.5 and fainter. It will be seen that the greater portion of such stars of large velocity are of absolutely faint

[1] *Astrophysical Journal*, **45**, 295, 1917.

magnitude, and that the total number of such stars is only about 3 per cent of the whole. Some such method appears to be excellent for using all data and justified, at least in principle. If, however,

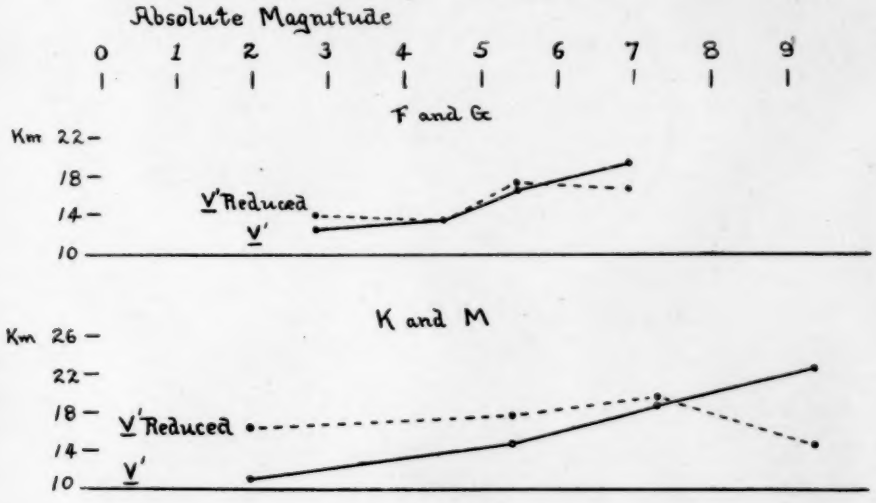

FIG. 1

TABLE III*

π	$\overline{M}$	No. of Stars	$\overline{V'}$	$\overline{V'}$ Reduced	$\underline{V'}$	$\underline{V'}$ Reduced
	F and G Stars					
			km	km	km	km
0".076.....	2.9	41	15.3	16.7	12.7	13.9
.082.....	4.47	35	16.8	17.0	13.4	13.5
.080.....	5.35	33	16.0	16.6	16.9	17.5
.094.....	6.92	33	21.2	18.8	19.2	16.9
	K and M Stars					
0".066.....	1.95	19	17.2	25.3	11.1	16.3
.081.....	5.38	38	25.7	30.7	14.8	17.8
.092.....	7.27	24	27.5	29.0	18.8	19.8
.149.....	9.34	27	28.4	18.5	22.2	14.5

* A bar above the symbol denotes the arithmetical mean. A bar below the symbol denotes the geometric mean.

the inclusion of such a small percentage of the whole has a material effect on the reality of any phenomena its inclusion in that particular case would seem not to be warranted. After the finding of the

peculiar effect in the two groups of large parallax, noted above, I tried the effect of limiting a test to the remaining 97 per cent as given in their values of $\overline{V}'$ in Table IV, where these large values had been excluded. The reduced values for the groups of large parallax are given in column 5 of Table III and the curve in Fig. 2. In neither of these does the evidence appear to be any stronger of a dependence upon M than upon π.

They conclude also[1] that "There seems to be little evidence of a decrease in velocity with distance for a constant absolute magni-

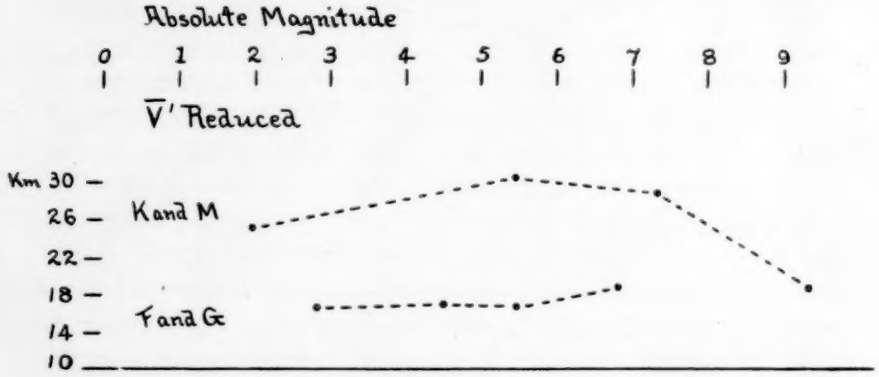

FIG. 2

tude, and it appears certain that the effect if present must be relatively slight." This statement probably refers more particularly to Table IV, but seems to me to be in conflict with the results of Tables V and VI, where strong progressions are shown in π for both spectral groups. Indeed the changes in π in these tables appear to be fully as well marked as the changes in M. These characteristics as well as an abrupt change at about the middle of the groups and differences in the small π-bright M and the large π-faint M stars are better shown in the curves in Figs. 3 and 4 which have been prepared from Table V. On account of the abrupt change mentioned and their different characteristics, the two portions of the curves were made discontinuous at the middle.

These curves show a striking similarity in all parts, a similarity which is undoubtedly significant. ***Both spectral groups show increases of velocity with increasing π and decreasing M for the small π***

[1] *Astrophysical Journal*, 45, p. 302.

and bright M groups and no certain change of velocity in the large π *and faint M groups.*

A direct interpretation of the mere fact of the progressions of π and M of Tables V and VI suggests that distance and absolute magnitude are equally concerned. The peculiarities of the curves of Figs. 3 and 4 are better satisfied, however, in my opinion by the conclusion that the dependence, whatever it is, is not a *continuous* function. This seems very clearly indicated by the curves. If

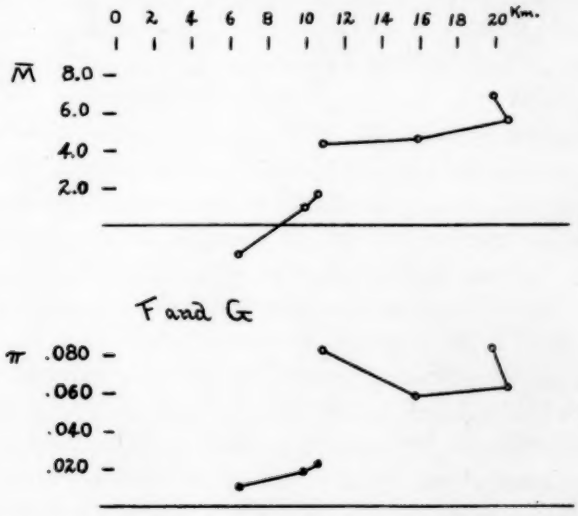

FIG. 3

such a discontinuity should be confirmed from more extensive data it would appear to be difficult to find an adequate explanation in a dependence wholly upon absolute magnitude. Some hypothesis in which distance is concerned, probably rather in the nature of regions, seems to me to offer a more satisfactory explanation of the observed facts. Taking all of these into consideration, I have no hesitancy in expressing the belief that distance in some manner plays a considerable part in the phenomena in question. If my hypothesis is correct the chief factor is distance or region, and the absolute magnitudes, spectral classes, and changes of velocity follow from the different conditions existing in the near and distant regions of our stellar system. This would explain also the very close relation of all of these factors.

The radial velocities of the stars appear to be so closely entangled with their magnitudes, spectral classes, distances, and streaming that the only hope of disentangling them completely seems to be by classifying the data simultaneously according to all of the factors concerned, including preferential motion. The considerable number of these factors necessitates so many subdivisions of the data

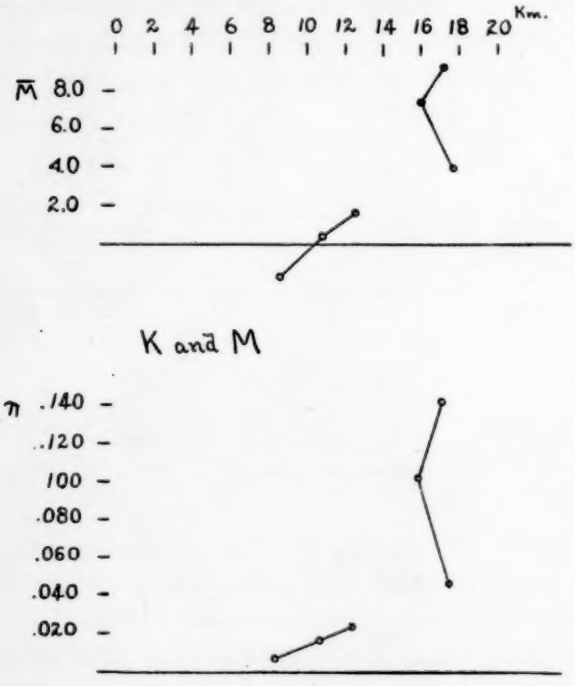

FIG. 4

as to make it desirable that the conclusion arrived at—that the dependence of velocity is primarily upon distance and streaming—should be confirmed by much more extensive data than is available at present.

Before going farther in explanation it is necessary to attempt to determine whether or not the smaller velocities for the early type, distant stars is a general phenomenon, i.e., spatial, or whether it may be explained by assuming that the line of sight may make a large angle with the direction of a stream. At present the only

evidence that I can discover which bears on this is that of the motions in the stars (the nearer ones) which show the strong preferential motion. This applies, of course, only by extrapolation.

If it is a case of stream-motion entirely and the direction of the line of sight with respect to the axis of the stream, the observer being in the stream in the one case (the near stars) and looking at the stream at a considerable angle, from a distance, in the distant stars, then the radial velocities at right angles to the stream-motion in the near stars (which show preferential motion) should be the same as the average radial velocities for the distant stars. An examination of Tables I and II shows a well-marked increase of ρ_1, (the minor axis) for the near stars.

So far as this data is competent, therefore, it indicates that the decrease of radial velocity in the distant stars is general and not due entirely to the direction which the line of sight makes with stream-motion. This appears to be independent of whatever other relations there may be between proper motion and radial velocity, so long as the larger proper-motion stars are also nearer.

The hypothesis based upon the assumption of finely divided cosmical matter in the relatively distant galactic regions, which has been suggested to account very largely for the observed differences of spectral type, also appears competent to explain the smaller velocities of the distant stars, whether this dependence is entirely upon distance or partially upon absolute magnitude and spectral class. This hypothesis may be stated briefly as follows:

The spectral condition of a star depends chiefly upon its size and mass and the external conditions of density of cosmical matter and relative velocities of star and matter. This matter is assumed to be most dense in the relatively distant regions in the direction of the galaxy.

The theory of such a change of velocity is that the cosmical matter acts as a resisting medium and *slows down* the higher velocities. This hypothesis points to a reversal of the direction of change of velocity; that instead of increasing from the early to the late types the velocities of the later type (and nearer stars) have been retarded. The operation is so obvious in principle that no more detailed explanation seems called for at this time.

No attempt is made to account for the present (higher) velocities of the nearer- and later-type stars, nor to explain the greater evolutionary changes in the motions of our stellar system. The present investigation is limited to a single, perhaps transient, phase of the great evolutionary process.

This hypothesis suggests the question, Have the stars grown up by accretion from very small beginnings, their velocities gradually slowing down during the process? If this is so, then we should expect the stars of small mass to have greater velocities than those of large mass, thus conforming to an absolute magnitude progression.

As far as the B-type stars, the reduction of velocity appears to be more or less systematic. We know little as yet of the radial velocities of the Wolf-Rayet stars and have too few velocities of the novae to make it possible to discuss these two classes with profit at present. But there are sufficient observations of the velocities of the planetary nebulae to show that the velocities of these bodies are undoubtedly large, perhaps even larger on the average than the later-type stars. These high velocities appear to be required by the hypothesis[1] put forth to account for these objects, viz., that the outburst was of unusual violence on a *single body*.

The large, irregular gaseous nebulosities appear to be nearly at rest with respect to the stellar system. But probably too few of them have been observed to serve as a sufficient basis for discussion as yet. It may not be out of place, however, to consider the cause on the assumption that low velocities should prove to be the rule. If the suggestion which has been made to account for these bodies should be confirmed, low general velocities might be expected. The suggestion was based upon the observed fact that many of the largest of these bodies have two or more bright stellar bodies more or less centrally located in the mass and was that the nebulosity had been thrown off in a collision or near-collision among these bodies. The resultant of such opposing velocities as here indicated would probably be low.

OBSERVATORIO NACIONAL ARGENTINO, CÓRDOBA
November 30, 1917

[1] "On the Cause Underlying the Spectral Differences of the Stars" (in press), *Astrophysical Journal*, 1918.

RADIAL VELOCITY AND ABSOLUTE MAGNITUDE

COMMENTS ON PROFESSOR PERRINE'S ARTICLE[1]

By WALTER S. ADAMS and GUSTAF STRÖMBERG

Professor Perrine has been so kind as to send us the manuscript of his article in advance of publication, and it will perhaps aid the reader to form a clearer idea of the points at issue if we comment briefly upon certain matters involved. The question is primarily one of the interpretation of the increase of radial velocity with proper motion. Perrine considers this as mainly an effect of distance from the sun, while we regard it as dependent upon the absolute magnitudes of the stars considered.

In Tables I and II of his article Perrine has divided the stars into two groups defined by proper motions less than 0".1 and greater than 0".1, respectively. Within these groups he arranges the stars according to apparent magnitude and gives the radial velocities along the axes of the ellipsoid. Of the stars in Tables I and II he says: "In the stars of small μ (Table I) the average μ is not given for the different groups of magnitudes, but it will probably not differ very greatly, as the same superior limit was adopted for all. . . . . In Table II the range of values of μ is considerable and is taken into account." As a result of this argument Perrine concludes that the grouping by apparent magnitude becomes a grouping according to relative absolute magnitude as well.

To this mode of treatment of the stars in Table I we are obliged to object seriously. The dispersion in distance among these stars is enormous, much greater than that for the stars in Table II, if we assume, as Perrine does, that distance is inversely proportional to proper motion. Since B stars are included by him in Table I, we may have a range of from 0".001 to 0".100 in the proper motion, or a range in distance of 100 to 1 on this basis. It would, in fact, be more correct to assume that the stars within the zone of proper motions larger than 0".1 are all at the same distance than to assume

[1] *Contributions from the Mount Wilson Solar Observatory*, No. 146.

this for the more distant stars without proof. It is therefore not possible to conclude that the grouping of stars according to apparent magnitude in Table I corresponds to a grouping according to absolute magnitude.

A second very natural objection to Perrine's method of applying to the radial velocities factors which will reduce all the stars to the same proper motion (and distance) is the nature of the consequences involved. If a group of stars with an average proper motion of 0".3 has one-half the average radial velocity of a group with a proper motion of 0".6, the velocity of a group with proper motion of 0".005 will be far less than any average observed radial velocity. It would be necessary to assume that the change of velocity with distance is greatest near the sun. This involves the remarkable assumption that the sun is a singular point in space (at the present time), and also it is opposed to Perrine's hypothesis that cosmical matter acts as a resisting medium, since he supposes this matter to lie in relatively distant galactic regions.

In a discussion of our results Perrine concludes that in the two groups of stars of types F and G, and K and M, which have the largest parallaxes, there is a "small but well-defined progression in the mean parallaxes of the groups in the same direction as the progression in velocity." This is quite true, but this progression corresponds to an exceedingly small range in distance when compared with that of the more distant stars. The two groups to which he refers have parallaxes larger than 0".05. If velocity is inversely proportional to distance within this comparatively narrow zone, we should have a velocity approaching zero at a relatively small distance, a conclusion which is contradicted by observation.

Perrine selects from our results only two groups of stars from a total of thirteen, since the parallaxes are most reliable in the case of the larger values. No doubt the distances of the nearer stars are known more accurately, but it is quite inadmissible to neglect the evidence afforded by the stars of small parallax. One of the principal advantages of the spectroscopic method of deriving parallaxes is that it is independent of their absolute values. Accordingly the relative determinations of distance, even among the stars

of smallest parallax, should be entitled to considerable weight. We find, however, that the average radial velocity for a group of over 200 stars with an average parallax of about 0".010 is practically the same as that for a similar group with a parallax of 0".050 of the same absolute magnitude.

Two other statements by Professor Perrine require a word of comment. His reference to Tables V and VI of our previous article as indicating increase of radial velocity with increase of parallax is due to a misconception of their purpose. Among the very distant stars we cannot observe stars of very low absolute magnitude, since such stars are apparently too faint. Accordingly Tables V and VI naturally show a progression of magnitude with increase of parallax and radial velocity. The attempt to separate the effect of distance from that of absolute magnitude is confined to the previous discussion, and the results are contained in Table IV.

As regards the effect of stream-motion upon our velocities, no attempt was made to eliminate it completely from our earlier results. There can, however, be no doubt that its influence was reduced greatly by our inclusion of velocities at right angles to the line of sight, and the confirmatory evidence obtained from these values is of considerable importance. An extended investigation of stream-motion as related to absolute magnitude has been made by Strömberg and is now in press. Its results are in agreement with our previous conclusions.

In general, it may be said that the principal cause of difference between Professor Perrine and ourselves is one of point of view. We have attempted in our discussion of stellar motions to find a fundamental relationship which would apply over the entire range of distance and absolute magnitude which has been investigated. For the reasons already given, it does not seem possible to conclude that a law making radial velocity inversely proportional to distance from the sun can hold over more than a very limited range. Outside of a comparatively narrow limit it would give values of the radial velocity much too high or too low, unless the law were modified greatly. Moreover, such a relationship would assign to the sun an extraordinary position in the stellar system, which we have no reason for believing that it holds. It is a star of very

moderate size and mass, and is known to lie at a great distance from the center of the galaxy. It would seem far more probable that any relationship between velocity and distance would be one in which the latter was measured from some much more fundamental reference point than our sun.

MOUNT WILSON SOLAR OBSERVATORY
February 1918

ORBIT OF THE SPECTROSCOPIC BINARY 32 θ_2 CYGNI

By J. B. CANNON

32 θ_2 Cygni ($\alpha = 20^h\ 12^m$, $\delta = +47°24'$; mag., 5.15; type G5) was announced a binary in *Lick Observatory Bulletins*, **4**, 96, from four measures giving a range of 30 km. Later in *Astronomische Nachrichten*, **198**, 409, 1914, Küstner published three measures of plates taken in 1908, 1909, and 1911, having a range of 40 km. The star was under observation here during the years 1914, 1915, 1916, and 1917, during which time 117 plates were taken.

The following determination of the elements must be considered as preliminary. The period is so long, 1170 days, that the measures were only beginning to repeat themselves at the end of the fourth year. More observations several years hence may change the period by some days, but the other elements as determined are probably fairly accurate. Five plates taken in 1908 helped in the determination of the period.

The wave-lengths adopted for the lines used are given in Table I.

TABLE I

4571.896	4395.461	4289.872
4549.646	4351.674	4282.858
4523.052	4340.705	4271.888
4501.865	4326.076	4215.906
4415.200	4304.891	4128.108
4404.042	4294.514	4101.653

The plates were grouped into twelve normal places found in Table II.

A curve running through the normal places has some similarity to that which would be given by the blending of the lines of the primary star by those of a secondary star, but the deviation from the elliptic form takes place unsymmetrically in such a way as to be hardly thus accountable. Also there appears to be no widening

of the lines at any place in the orbit and it seems as though the irregularities in the curve are due to actual variations in the motion of the light-giving body. It was therefore attempted to run an elliptical curve as closely as possible through the normal places and

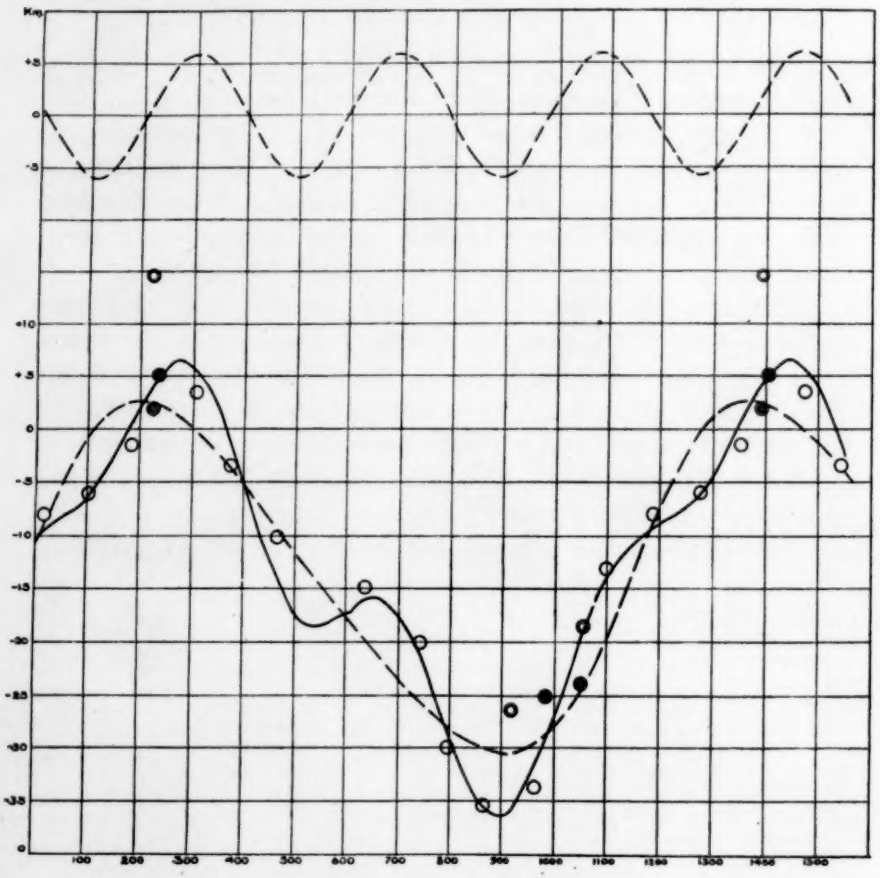

FIG. 1.—Radial velocity-curve of 32 θ_2 Cygni

apply a secondary correction to this by introducing a circular curve of one-third the period (Fig. 1).

A least-squares solution was carried through, which resulted in slight corrections to the elements and a reduction of the value of Σpvv from 75 to 60. The preliminary and final values of the

TABLE II

No.	Julian Day	Phase	Velocity	Weight	Residual
			km		km
1	2420510.835	795.708	−30.1	4	−1.6
2	543.658	867.018	−35.4	3	+0.2
3	495.178	964.788	−33.6	1	−2.1
4	629.599	1099.209	−13.1	1.5	+0.8
5	716.627	16.237	− 7.8	2	+1.3
6	801.337	100.947	− 5.9	2	±0.0
7	882.701	182.311	− 1.4	1	−2.1
8	1007.588	307.198	+ 3.7	1	−1.9
9	1076.982	376.592	− 3.4	1.5	−1.7
10	1166.664	466.274	−10.1	1	+4.5
11	0272.465	635.711	−14.9	2	+1.6
12	1443.212	742.822	−20.1	2	+1.6

elements are given below. Probable errors were computed and are appended to the final values.

Element	Preliminary	Final
P	1170 days	1170 days
e	0.2	0.182 ± 0.053
ω	280°	$281°.05 \pm 4°.8$
K	16 km	16.64 ± 0.93 km
T	2420700.25 J.D.	$2420700.39 \pm 18^d.7$
γ	−14.6 km	-14.35 ± 0.65 km
K_1	6 km	5.86 ± 0.90 km
T_1	2420515 J.D.	$2420515.821 \pm 12^d.4$
$a \sin i$	.....	263250000 km
$\frac{m_1^3 \sin^3 i}{(m+m_1)^2}$	.....	0.53 ⊙

To explain the curves one would have to consider the system as consisting of a light-giving body revolving about another body in a circular orbit in 390 days, and these two about a third in an elliptic orbit in 1170 days (Fig. 2).

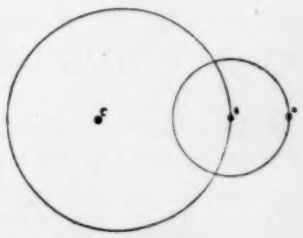

FIG. 2

In Fig. 1 the single circles represent the Ottawa normal places, the double circles Küstner's observations, and the triple circles the Lick observations.

DOMINION OBSERVATORY, OTTAWA, CANADA
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ON THE ORBIT OF THE SPECTROSCOPIC BINARY ALPHA PHOENICIS

By JOSEPH LUNT

The variable velocity of this star ($\alpha = 0^h21^m.3$, $\delta = -42°51'$ [1900]; magnitude = 2.44; type K) was announced in *Lick Observatory Bulletin*, No. 75 (**3**, 110) by Wright. Five plates had then been secured in 1903–4 showing a range of 5.5 km. The star has been under observation here from November 27, 1903, to December 28, 1914, giving an observed range of 11.2 km.

The last plate showed that a cycle had been completed, and as no further observations or orbit have been published, so far as the writer is aware, it seems desirable to publish the results obtained and to give a preliminary orbit which fits the present observations satisfactorily. It is regretted that observations were interrupted after July 18, 1912, when the maximum velocity was approaching, and the peak of the curve was not observed. A single plate on December 28, 1914, however, has served to indicate the period, and the following orbit was derived by graphical methods.

$$P = 3880 \text{ days}$$
$$= 10.62 \text{ years}$$
$$T = \text{J.D. } 2416185$$
$$= 1903 \text{ March } 11$$
$$\omega = 20°$$
$$e = 0.32$$
$$K = 6.0 \text{ km}$$
$$V_0 = +75.76 \text{ km}$$
$$a \sin i = 308{,}000{,}000 \text{ km}$$

α Phoenicis is a remarkable star in some respects. If we include visual binaries, it appears to be second only to σ Puppis[1] in

[1] See *Astrophysical Journal*, **44**, 260, 1916.

velocity of the center of mass, to β Capricorni in size of orbit, and to Polaris in length of period. The data compare as follows.

	P	e	$a \sin i$	K	V_0
	Years		km	km	km
α Phoenicis	10.62	0.32	308,000,000	6.0	+75.8
σ Puppis	0.706	0.20	62,570,000	18.0	+88.5
Polaris*	11.9	0.35	166,800,000	2.98	−14.8
β Capricorni	3.77	0.44	377,000,000	22.2	−18.8

* Center of mass of short-period binary system.

The orbit given indicates that the radial velocity is now at a minimum (January 1918) and increasing to a maximum in February 1924. The velocities given in the column of remarks of Campbell's second list of binaries, viz., 1903.7, +80 and 1907.8, +70, are half a kilometer and a kilometer lower, respectively, than the present orbit demands but agree within the errors of observation.

Probably other observations have been made at Santiago and it will be of interest to fill in the lacunae.

Table I shows the observations compared with the computed velocities. Of the 44 plates considered, including 5 Santiago plates, 25 give residuals from 0.0 to 0.5; 14 from 0.6 to 1.0; and 5 above one kilometer. The measures were made by the writer with the Hartmann spectrocomparator and solar standards (daylight spectra) after 1908, and α Bootis, Plate No. 1191, before the end of that year.

The plates marked * were also measured by Dr. Halm, using α Tauri, Plate No. 2145, as standard, and the means of his measures and the writer's were adopted.

The upper scales of years and Julian days in the figure refer to the epoch of observation and the lower ones to the curve regarded as an ephemeris commencing with the periastron of October 24, 1913. The black-centered circles refer to means of groups of plates. The triangles represent Santiago measures.

TABLE I

Plate No.	Date	J. D. 2400000 +	Phase Days	Radial Velocities		Residuals O−C
				Observed	Computed	
				km	km	km
—‡	1903 Sept. 15*	16373	188	+80.7	+80.5	+0.2
—‡	Oct. 1‖	389	204	79.0	80.2	−1.2
—‡	5*	393	208	79.8	80.2	−0.4
§ 615	Nov. 27	446	261	79.3	79.3	0.0
§ 616	Dec. 2	451	266	79.5	79.3	+0.2
§ 617	12	461	276	80.0	79.1	+0.9
§ 618	15	464	279	81.8**	79.1	+2.7
§ 619	17	466	281	78.9	79.0	−0.1
§—‡	1904 Aug. 3¶	696	511	75.2	75.8	−0.6
§ 697	26	719	534	74.8	75.5	−0.7
§—‡	Sept. 10	734	549	75.5†	75.3	+0.2
§ 915	1906 May 29	17360	1175	70.5	71.4	−0.9
§1021	Oct. 22	506	1321	71.8	71.1	+0.7
§1022	23	507	1322	71.0	71.1	−0.1
§1287	1907 Nov. 5	885	1700	73.3**	71.1	+2.2
§1294	8	888	1703	71.6	71.1	+0.5
§1348	Dec. 17	927	1743	70.4	71.1	−0.7
1824*	1908 Aug. 13	18167	1982	72.8	71.4	+1.4
1871*	Sept. 1	186	2001	70.6	71.5	−0.9
1881*	3	188	2003	72.7	71.5	+1.2
1885*	4	189	2004	71.7	71.5	+0.2
2081*	Dec. 17	293	2108	72.1	71.7	+0.4
2494	1909 Sept. 8	550	2365	72.8	72.6	+0.2
2535	Oct. 14	594	2409	72.3	72.7	−0.4
2548	22	602	2417	73.2	72.8	+0.4
2553	29	609	2424	71.8	72.8	−1.0
2584*	Nov. 17	628	2443	72.9	72.9	0.0
2762	1910 July 26	879	2694	73.4	74.1	−0.7
2765	27	880	2695	73.9	74.1	−0.2
2768	31	884	2699	74.1	74.1	0.0
2837	Oct. 4	949	2764	74.6	74.5	+0.1
2842	7	952	2767	73.5	74.5	−1.0
3220	1911 July 9	19227	3042	76.2	76.6	−0.4
3225	16	234	3049	76.5	76.6	−0.1
3308	Oct. 10	320	3135	76.7	77.4	−0.7
3355	Nov. 16	357	3172	77.7	77.8	−0.1
3378	Dec. 6	377	3192	78.8	78.0	+0.8
3387	9	380	3195	77.9	78.0	−0.1
3399	19	390	3205	79.0	78.0	+1.0
3651*	1912 June 17	571	3386	80.0	80.0	0.0
3665*	23	577	3392	79.2	80.1	−0.9
3682*	July 1	585	3400	80.0	80.2	−0.2
3720*	18	602	3417	80.2	80.3	−0.1
4467*	1914 Dec. 28	20495	0455	76.5	76.8	−0.3

* Curtis. † Correction +1.1 km applied. ‡ D. O. Mills Expedition.
§ From *Annals of the Cape Observatory*, 10, 52. ‖ Palmer. ¶ Wright. ** Weight ½.

Royal Observatory, Cape of Good Hope
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Minor Contributions and Notes

DESIRABILITY OF OBSERVATIONS OF THE SOLAR ROTATION AT THE TIMES OF ECLIPSE

In the lines of spectra from the solar limb there is blending of such nature as to affect the measurements of the velocity of the sun's rotation from the displacements of these lines. The magnitude of the effect varies from line to line. Various factors contribute to the blending of spectra.[1] The factors operating between the moon and the point of observation are nearly eliminated just before and after totality, when only the limb under observation is visible. To investigate the problem a powerful spectroscope (producing spectra of scale about 1 mm to 1 A) is necessary and a large image of the sun preferred. The region of the *b*-group is suggested because the effect of blending is known for the *b*-lines and others near them, and because iodine and chlorine comparison spectra are available. It is very desirable that such observations be made at the eclipse of June 8, so that terrestrial sources of blended spectra may be removed in order to investigate the solar and interplanetary sources. The observations need not interfere with those during the precious seconds of totality.

At least one party is planning[2] to make determinations of the speed of rotation of the corona from displacements of the coronal lines at the time of total eclipse on June 8. In view of the fact that Deslandres, Campbell, and Bosler have obtained values of this speed of from 3 to 4 km per second, it is very important that every precaution be taken to keep check on the errors that beset such measurements. In this regard I would suggest that the spectroscope be used to determine the velocity of rotation of the reversing layer by observations of the east and west limbs of the

[1] *Journal of the Royal Astronomical Society of Canada*, **10**, 201–219, and 345–357, 1916; *Astrophysical Journal*, **44**, 177–189, and 198–199, 1916.

[2] Edwin B. Frost, *Popular Astronomy*, **26**, 110, 1918.

sun before and after totality of eclipse, using a comparison spectrum and employing precisely the same method in determining the velocity from the coronal lines. In this way one can judge as to the reliability of the latter. If it is impossible to use an arc or a spark comparison spectrum, the stronger iodine absorption lines or the narrow strips of continuous spectrum between them may be found suitable in the neighborhood of λ 5303.

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